

NASA Contractor Report 3787

Theory and Modeling of Atmospheric Turbulence

Part I: September 1, 1981-August 31, 1982

C. M. Tchen

CONTRACT NAS8-34622
FEBRUARY 1984

LOAN COPY: RETURN TO
AFWL TECHNICAL LIBRARY
KIRTLAND AFB, N.M. 87117

NASA

NASA
CR
3787
c.1

TECH LIBRARY KAFB, NM
0062344



NASA Contractor Report 3787

Theory and Modeling of Atmospheric Turbulence

Part I: September 1, 1981-August 31, 1982

C. M. Tchen

*The City College Research Foundation
New York, New York*

Prepared for
George C. Marshall Space Flight Center
under Contract NAS8-34622



National Aeronautics
and Space Administration

**Scientific and Technical
Information Office**

1984

FOREWORD

Two approaches for the modeling of turbulence exist. One is to start from an open hierarchy of profiles and higher order correlations. The other is to formulate a closed sequence of transport processes: evolution of mean profiles, eddy transport coefficients (viscosity, diffusivity, and damping or amplification rate), and relaxation or memory loss for the approach of the transport coefficients to equilibrium. The former approach uses an ambiguous closure, or an arbitrary hypothesis of the length scale of mixing. The latter approach is based on a theory of transport and the analysis of the spectral structure of turbulence.

The author follows the latter approach and develops a new kinetic method. This intact report includes research reported quarterly during the first year of effort, September 1981 through August 1982, with a second report to follow, which will include research for September 1982 through August 1983.

ACKNOWLEDGEMENTS

The author wishes to acknowledge the support for this research of Mr. John S. Theon and Dr. Robert Curran of NASA Headquarters, Washington, D. C., and the technical direction of Dr. George H. Fichtl and Mrs. Margaret B. Alexander of the Atmospheric Sciences Division, Systems Dynamics Laboratory, Marshall Space Flight Center, Alabama.

TABLE OF CONTENTS

QUARTER ONE: Kinetic Basis of Cascade Transfer in Turbulence

| | <u>Page</u> |
|--|-------------|
| Abstract. | A-1 |
| I. Introduction | A-2 |
| II. Micro-Dynamical State of Turbulence in the Configuration and Phase Spaces | A-6 |
| III. The BBGK Hierarchy of Turbulence | A-11 |
| IV. The Scaling of Fluctuations | A-15 |
| V. Kinetic Equation of Turbulence and Kinetic Theory of Transition. | A-20 |
| VI. Transport Theory of Diffusivity | A-30 |
| VII. Kinetic Basis of the Cascade Transfer | A-35 |
| VIII. Conclusions | A-40 |
| IX. Acknowledgment | A-42 |
| X. References | A-43 |

QUARTER TWO: Kinetic Theory of Turbulent Transfer with Double Memory Loss

| | |
|--|------|
| Abstract. | B-1 |
| I. Introduction | B-2 |
| II. The Scaled Kinetic Equation of Turbulence | B-5 |
| III. The Lagrangian-Eulerian Transformation in the Phase Space, The Loss and Cutoff of Memory | B-9 |
| IV. Cascade Transfer with Double Memory Loss | B-16 |
| V. Eddy Viscosity and Collisionless Damping | B-21 |

TABLE OF CONTENTS (cont)

| | Page |
|---|------|
| VI. Derivation of the Kilmogoroff Law from the Scaled Kinetic Equation | B-28 |
| VII. Discussion | B-31 |
| Acknowledgement. | B-35 |
| References | B-36 |
| QUARTER THREE: Group Scaling Theory for the Enstrophy Transfer in Two-Dimensional Turbulence | |
| Abstract. | C-1 |
| I. Introduction | C-3 |
| II. Scaled Vorticity Equation of Turbulence. | C-5 |
| III. Transport Phenomena in Turbulence. | C-10 |
| IV. Double Memory Loss | C-17 |
| V. Transfer Function | C-19 |
| VI. Inertia Spectrum | C-20 |
| VII. Discussions and Conclusions. | C-21 |
| Appendix | C-24 |
| Acknowledgement. | C-31 |
| References | C-31 |
| QUARTER FOUR: A Group-Kinetic Theory of Turbulent Collective Collision | |
| Abstract. | D-1 |
| I. Introduction | D-3 |
| II. Microdynamical State of Turbulence | D-8 |
| III. Group-Scaling Procedure. | D-11 |
| IV. Turbulent Collisions and Memories. | D-14 |

TABLE OF CONTENTS (cont)

| | Page |
|--|------|
| V. Kinetic Equation of the Transition Probability. | D-22 |
| VI. Determination of the Collisions by Means of the Transition Probability. | D-29 |
| VII. Shielding and Enhancement of the Collision. | D-33 |
| VIII. Summary and Discussions. | D-37 |
| Acknowledgment | D-43 |
| References | D-44 |

CHAPTER A: QUARTER ONE

Kinetic Basis of Cascade Transfer in Turbulence

C. M. Tchen

The Graduate Center and The City College of the City
University of New York, N.Y. 10031

ABSTRACT

Among the transport functions which characterize the evolution of a turbulent spectrum, the cascade transfer is the only function which describes the mode-coupling as the result of the nonlinear hydrodynamic state of turbulence. A kinetic theory combined with a scaling procedure is developed, to derive a kinetic equation of velocity distribution and the transition equations of path perturbations. These equations permit to formulate a transport theory of turbulence to investigate the eddy diffusivity, the eddy viscosity and the cascade transfer. Circumstances under which the transfer may appear in the form

of a direct cascade or a reverse cascade are considered.

I. INTRODUCTION

The Navier-Stokes equation of motion, with a pressure gradient and a buoyancy as driving fields, can be used to describe the micro-dynamical state of turbulence and to serve as the basis of a statistical theory. But the nonlinearities in the inertia and the driving fields will confront us from the start of the analysis. Various statistical methods have been proposed in the literature but were not too successful in solving the closure of the hierarchy of equations. By changing the velocity function into a "particle" velocity as an independent random variable, the kinetic method is relieved of this diffi-

culty. What remains is a nonlinear interaction between fluid and "particle", with a diffusivity as an integral operator to represent the memory in the non-Markoff process.

The kinetic theory of turbulence presents the following three problems: (a) the kinetic equation of turbulence, (b) the transport theory of turbulence for the determination of the eddy diffusivity, (c) the kinetic basis of the cascade transfer. In the derivation of a kinetic equation, it is important not to fall into the Bogoliubov hierarchy.^{1,7} Its closure has not been as successful in fluids as in plasmas.⁸ This difficulty may have prevented further progress in the kinetic theory of fluid turbulence. We introduce a "scaling procedure" to avoid this hierarchy and the involvement of a system of equations of singlet and pair distribution functions.

The eddy diffusivity which characterizes the kinetic equation is defined as the time integration of the Lagrangian correlation of field fluctuations. We develop a transport theory for the transformation of a Lagrangian correlation into an Eulerian correlation. This relationship, or the time-space transformation, has attracted many investigators by using the hypotheses of independence and normality.⁹ The correction has also been evaluated.^{10,11} The problem referred to a diffusion model, i.e. to a micro-dynamical state without a driving field. This model would exclude the more realistic aspects of turbulence, as presented by the Navier-Stokes turbulence, the shear turbulence and the turbulent motions in a stratified medium. We feel it important to find a generalization that incorporates the driving field. This generalization brings up

new difficulties. Our kinetic theory of turbulent transport helps in switching the driving field to a new role of advection in the phase space and in prescribing probabilities of transition to replace the above hypotheses of independence and normality.

In order to be qualified as a transport property, the diffusivity has to reach a statistical equilibrium within a finite time, called the relaxation time. This finite time is obtained, when a memory-loss can be found. The scaling procedure selects and organizes the necessary mechanism of memory-loss by the turbulent dissipations.

The mode-coupling governs the transfer of modes across the spectrum in both directions toward the high as well as the low wavenumbers. It appears in the form of the moment of

the fluid-particle interaction, in analogy with the nonlinear Landau damping in plasma turbulence. Formally, the fluid-particle interaction can be expressed by a series of high order moments, as we know that a velocity distribution could be expanded in this way. It is nevertheless an insufficient way of representation. This indicates that, with these limitations, a pure continuum method on its own may not be successful in analyzing the mode-coupling, since it had failed in describing the Landau damping.

II. MICRO-DYNAMICAL STATE OF TURBULENCE IN THE CONFIGURATION AND PHASE SPACES

The hydrodynamical equations of the motion of an incompressible fluid are the equation of momentum

$$(\partial_t + \hat{L}) \hat{\underline{u}} = \hat{\underline{E}}, \quad \partial_t \equiv \partial/\partial t, \quad (1a)$$

and the equation of continuity

$$\nabla \cdot \hat{\underline{u}} = 0, \quad (1b)$$

where the evolution of the total fluid velocity $\hat{\underline{u}}(t, \underline{x})$ under the force $\hat{\underline{E}}(t, \underline{x})$ is governed by a differential operator

$$\hat{L}(t, \underline{x}) \equiv \hat{\underline{u}} \cdot \nabla - \nu \nabla^2, \quad (2)$$

and ν is the kinematic viscosity of the fluid. The force

$$\hat{\underline{E}} = -\frac{1}{\rho} \nabla p \quad (3)$$

may be due to the gradient of pressure \hat{p} in a fluid of density

ρ , or may take a more complicate form involving the temperature difference and the buoyancy force as in a stratified medium. In all cases, the force can be written in the differential form

$$\nabla \cdot \hat{\underline{E}}(t, \underline{x}) = \nabla \nabla : \hat{\underline{u}}(t, \underline{x}) \hat{\underline{u}}(t, \underline{x}) \equiv \hat{\lambda}(t, \underline{x}) \quad (4a)$$

or in the integral form

$$\hat{E}(t, \underline{x}) = g(t, \underline{x}/\underline{x}') \{ \hat{\lambda}(t, \underline{x}') \} \quad (4b)$$

where

$$g(t, \underline{x}/\underline{x}') \{ \} \equiv -\nabla \frac{1}{4\pi} \int d\underline{x}' \frac{1}{|\underline{x} - \underline{x}'|} \{ \} \quad (5a)$$

is Poisson's integral operator, such that

$$g(t, \underline{x}/\underline{x}') \{ \hat{\lambda}(t, \underline{x}') \} = -\nabla \frac{1}{4\pi} \int d\underline{x}' \frac{1}{|\underline{x} - \underline{x}'|} \hat{\lambda}(t, \underline{x}') \quad (5b)$$

Both forms result from applying the condition of incompressibility (1b) upon (1a). Here and in the following the integration is understood to extend from $-\infty$ to ∞ .

The hydrodynamical equations (1a) and (1b)

can be considered as describing the micro-dynamical state of turbulence in the configuration space t, \underline{x} . This can be transformed into the phase space $t, \underline{x}, \underline{v}$, by introducing a

δ - function

$$N(t, \underline{x}, \underline{v}) \equiv \delta[\underline{v} - \hat{\underline{u}}(t, \underline{x})] \quad (6)$$

which satisfies the equation of evolution

$$(\partial_t + \hat{L})N(t, \underline{x}, \underline{v}) = 0, \quad (7)$$

with a differential operator

$$\hat{L}(t, \underline{x}, \underline{v}) \equiv \underline{v} \cdot \underline{\nabla} - \gamma \nabla^2 + \hat{\underline{E}} \cdot \underline{\partial}, \quad \underline{\partial} \equiv \partial / \partial \underline{v} \quad (8)$$

in the phase space. The force can again be written in the inte-

gral form:

$$\hat{\underline{E}}(t, \underline{x}) = \underline{g}(t, \underline{x} / \underline{x}', \underline{v}') \left\{ n(\underline{x}', \underline{v}') N(t, \underline{x}', \underline{v}') \right\} \quad (9)$$

with Poisson's integral operator

$$\underline{g}(t, \underline{x} / \underline{x}', \underline{v}') \equiv - \underline{\nabla} \cdot \frac{1}{4\pi} \iint d\underline{x} \cdot d\underline{v}' \frac{1}{|\underline{x} - \underline{x}'|} \quad (10a)$$

and the source function

$$\begin{aligned} n(\underline{x}', \underline{v}') &\equiv \underline{\nabla} \underline{\nabla} : \underline{v} \underline{v} \\ &= (\underline{v} \underline{\nabla})^2 \end{aligned} \quad (10b)$$

in the phase space. ¹²

The micro-dynamical state in the phase space (7) possesses certain important advantages over that in the configuration space (1a) in eliminating the nonlinearities. This is done by changing the role of the velocity as a random function into the role as an independent random variable. By the same token, the inhomogeneous equation of micro-evolution in the configuration space is transformed into a homogeneous equation in the phase space. We shall fully utilize these advantages in developing our statistical theory of turbulence. Of course, the micro-dynamics should still keep a nonlinear form in the phase space, as seen from $\hat{E} \cdot \delta N$ in (7). Fortunately the treatment of this nonlinearity can be delayed by considering \hat{E} to be provisionally known. After we have closed the transport equations, we can pick up the nonlinear relationship (4a) which then will be in its uncomPLICATE form of a constitutive relation.

The equivalence between the two spaces (1) and (7), can be easily verified by taking the moments

$$\int d\underline{v} N = 1, \quad \int d\underline{v} \underline{v} N = \hat{u}(t, \underline{x}) . \quad (11)$$

of (7). It is then seen that the equation (7) in the phase space actually corresponds to the simultaneous system of dynamical equations (1a) and (1b) in the configuration space.

III. THE BBGK HIERARCHY OF TURBULENCE

If \underline{v} is a stochastic variable by the stochastic behavior of $\hat{u}(t, \underline{x})$, the function (6) can be called a distribution function, denoted by

$$N(t, \underline{x}, \underline{v}) = \hat{f}(t, \underline{x}, \underline{v}) , \quad (12)$$

retaining nevertheless its stochastic character, with a

micro - dynamical state (7). The function (12) and the evolution (7) contain fluctuations with all the minute details which are unnecessary in a statistical study. A coarse-graining procedure which eliminates all stochastic fluctuations is the ensemble average

$$\bar{A} = \langle \rangle \quad (13)$$

called "global average". A fluctuating function subject to this average becomes deterministic, e.g.

$$\bar{A} N(t, \underline{x}, \underline{v}) = \bar{f}(t, \underline{x}, \underline{v}) , \quad (14)$$

and the deviation from the average is the fluctuation

$$f - \bar{f} = \tilde{f} , \quad (15a)$$

or, in terms of the operator notations,

$$1 - \bar{A} = \tilde{A} , \quad (15b)$$

where "1" is the "unit operator", \bar{A} is the average operator,

and \tilde{A} is the fluctuation operator.

Upon applying the average operator \bar{A} to the equation

(7) of micro - dynamics, we get

$$\begin{aligned} (\partial_t + \hat{L}) \bar{f}(t, \underline{x}, \underline{v}) &= -\bar{A} \tilde{E}(t, \underline{x}) \cdot \underline{\partial} \tilde{f}(t, \underline{x}, \underline{v}) \\ &= -\underline{\partial} \cdot \underline{g}(t, \underline{x}/\underline{x}', \underline{v}') \left\{ n(\underline{x}', \underline{v}') \langle \tilde{f}(t, \underline{x}', \underline{v}') \tilde{f}(t, \underline{x}, \underline{v}) \rangle \right\} \end{aligned} \quad (16a)$$

or

$$(\partial_t + \bar{L}) \bar{f}(t, \underline{x}, \underline{v}) = -\underline{\partial} \cdot \underline{g}(t, \underline{x}/\underline{x}', \underline{v}') \left\{ n(\underline{x}', \underline{v}') \bar{f}(t, \underline{x}', \underline{v}'; \underline{x}, \underline{v}) \right\}, \quad (16b)$$

when use is made of (9) and of the notation

$$\bar{f}(t, \underline{x}', \underline{v}'; \underline{x}, \underline{v}) = \langle \tilde{f}(t, \underline{x}', \underline{v}') \tilde{f}(t, \underline{x}, \underline{v}) \rangle. \quad (17)$$

The functions (14) and (17) are called the singlet and the doublet distributions, respectively.

In (16b) we see that the evolution of the singlet distribution depends on the doublet distribution, and that the latter evolution will expectedly depends on the triplet

distribution

$$\bar{f}(t, \underline{x}, \underline{v}; \underline{x}', \underline{v}'; \underline{x}, \underline{v}) \equiv \langle \tilde{f}(t, \underline{x}, \underline{v}) \tilde{f}(t, \underline{x}', \underline{v}') \tilde{f}(t, \underline{x}, \underline{v}) \rangle, \quad (18)$$

and the sequence continues to form the BBGK hierarchy. This hierarchy is well known in the kinetic theory of plasma turbulence, and is seen here to reappear in fluid turbulence with the difference that $\underline{n}(\underline{x}, \underline{v})$ is defined by (10b) in an incompressible fluid, while $\underline{n}(\underline{x}, \underline{v}) = 1$ in plasmas. 1-4,8

The closure of the hierarchy constitutes a difficult problem. It has been attempted by finding a small parameter in plasmas, but such a parameter is lacking in fluids. This may explain why so little progress has been made in fluid turbulence by the kinetic approach. The analysis of the spectral structure of turbulence requires the Fourier transformation of a correlation function, and therefore the second

moment of the doublet distribution. Thus this kinetic problem calls for the determination of both distribution functions. This obviously would pose a tremendous task. We shall devise a kinetic method, in which we can avoid the BBGK hierarchy and the doublet distribution function. To this end, we shall resort to the scaling procedure of fluctuations, which we shall describe in the following.

IV. THE SCALING OF FLUCTUATIONS

The BBGK hierarchy which begins with the singlet distribution (16) does not give any statistical information unless the doublet distribution is known. This means that a minimum of two equations of the closed hierarchy will be

needed to describe the fluctuations statistically. In order to avoid the BBGK hierarchy completely, we shall scale the fluctuation

$$\tilde{\underline{E}} = \underline{E}^{(o)} + \underline{E}' \quad (19)$$

into a macro-group $\underline{E}^{(o)}$ and a micro-group \underline{E}' .

The scaling is performed by writing the fluctuation operator

$$\tilde{\underline{A}} = \underline{A}^{(o)} + \underline{A}' \quad (20a)$$

into two components $\underline{A}^{(o)}$, \underline{A}' , which select the macro-group and the micro-group, respectively, and are called "scaling operators". Since (15b) is known as the "Reynolds decomposition" in turbulence, we may consider the scaling (20a) as a simple extension of this decomposition.

For the sake of convenience, we introduce

$$A_o = \bar{A} + A^{(o)}, \quad (20b)$$

so that

$$A' = 1 - A_o. \quad (20c)$$

The two groups $\underline{E}^{(o)}$ and \underline{E}' have their durations of correlation $\tau_c^{(o)}$ and τ_c' , with the inequalities

$$\tau_c^{(o)} > \tau_c', \quad (21)$$

stating that the micro-group \underline{E}' is more random than the macro-group $\underline{E}^{(o)}$. Their respective portions of the energy

$$\frac{1}{2} \langle \underline{E}^{(o)2} \rangle = \int_0^k dk' S(k'), \quad \frac{1}{2} \langle E'^2 \rangle = \int_k^\infty dk' S(k') \quad (22a)$$

are clearly separated by the wavenumber variable k in the spectral distribution $S(k)$ of \underline{E} - fluctuations, although the individual groups $\underline{E}^{(o)}$ and \underline{E}' may overlap in their Fourier compositions. Clearly, the two portions add to

$$\frac{1}{2} \langle \underline{E}^2 \rangle = \int_0^{k=\infty} dk' S(k') . \quad (22b)$$

Similarly, we scale the velocity fluctuation

$$\underline{\tilde{u}} = \underline{u}^{(o)} + \underline{u}' \quad (23)$$

into a macro-velocity $\underline{u}^{(o)}$ and a micro-velocity \underline{u}' of energies

$$\frac{1}{2} \langle \underline{u}^{(o)2} \rangle = \int_0^k dk' F(k'), \quad \frac{1}{2} \langle \underline{u}'^2 \rangle = \int_k^\infty dk' S(k') \quad (24a)$$

in the spectral distribution $F(k)$ of \underline{u} - fluctuations.

The two portions also add to

$$\frac{1}{2} \langle \underline{u}^2 \rangle = \int_0^{k=\infty} dk' F(k) . \quad (24b)$$

The same scaling applies to the distribution function

$\tilde{f}(t, \underline{x}, \underline{v})$ in the superposition

$$\tilde{f} = f^{(o)} + \tilde{f}' \quad (25)$$

of a macro-distribution $f^{(o)}$ and a micro-distribution f' .

It is to be remarked that the scaling permits the

derivation of the spectral distributions by simply differentiating (22a) and (24a) with respect to \hbar , while this was not the case with the non-scaled energies (22b) and (24b). Thus our statistical problem amounts to finding a kinetic equation of the scaled singlet distribution only, instead of both the singlet and the doublet distributions in the unscaled treatment of the BBGK hierarchy.

It is to be stipulated that the decomposition into the scaled groups (20) differs from the decomposition into the Fourier components or a group of such components. The latter decomposition is a mathematical transformation without adding any physical concept to the treatment, while the former decomposition injects a scaling procedure of coarse-graining to the stochastic process by the operator $A^{(0)}$ or the operator

$A' = \tilde{A} - A^{(0)}$. Consequently, the scaling procedure distinguishes between a macro-process representative of the evolution of a macro-field and a micro-process representative of the transport properties as shaped by the more random micro-fluctuations in the medium in which the macro-field is propagating. Well defined probabilities of transition, that are self-consistent with the once given micro-dynamical state of turbulence, will define the scaling operators $A^{(0)}$ and A' .

V. KINETIC EQUATION OF TURBULENCE

The micro-dynamical state of turbulence is described by the dynamical state (7) in the phase space. We apply the scaling operators $A^{(0)}$ and A' and decompose (7) into the system

$$A^{(0)}(\partial_t + \hat{L}) \hat{f}_0(t, \underline{x}, \underline{v}) = - A^{(0)} \partial \cdot \underline{E}' f' \quad (26a)$$

$$(\partial_t + A' \hat{L}) f'(t, \underline{x}, \underline{v}) = - \underline{E}' \cdot \partial (\bar{f} + f^{(0)}), \quad (26b)$$

which describes the evolution of the macro-distribution under the statistical effects of the more random micro-fluctuations $\underline{E}' f'$, while the f' fluctuations are excited by the gradients of the mean and the macro-distributions, according to (26). All operators including the scaling operators $A^{(0)}$, A' and the differential operator L refer to all functions which follow them.

If we disregard the group-coupling between $f^{(0)}$ and f' and merely express \underline{E}' in terms of f' according to (9) by A' , we would again fall into the BBGK hierarchy in the form:

$$\begin{aligned} A^{(0)}(\partial_t + \hat{L}) \hat{f}_0(t, \underline{x}, \underline{v}) &= -A^{(0)} \partial \cdot g(t, \underline{x}/\underline{x}', \underline{v}') \left\{ n(\underline{x}', \underline{v}') f'(t, \underline{x}', \underline{v}') f'(t, \underline{x}, \underline{v}) \right\} \\ &= -\partial \cdot g(t, \underline{x}/\underline{x}', \underline{v}') \left\{ n(\underline{x}', \underline{v}') f^{(0)}(t, \underline{x}', \underline{v}'; \underline{x}, \underline{v}) \right\}. \end{aligned}$$

with

$$f^{(0)}(t, \underline{x}', \underline{v}'; \underline{x}, \underline{v}) \equiv A^{(0)} f'(t, \underline{x}', \underline{v}') f'(t, \underline{x}, \underline{v})$$

by definition of the scaled doublet distribution function.

In order to avoid this hierarchy which is equally undesirable, we call on the group-coupling between the two equations

(26a) and (26b). For this purpose, we rewrite (26b) in the form

$$(\partial_t + \hat{L}) f' = -E' \cdot \partial (\bar{f} + f^{(0)}) - A_0 \hat{L} f' \quad (27)$$

by the use of (20c). We integrate to find

$$\begin{aligned} f'(t, \underline{x}, \underline{v}) &\equiv A' \hat{f}(t, \underline{x}, \underline{v}) \\ &= - \int_0^t d\tau U(t, t-\tau) E'(t-\tau) \cdot \partial [\bar{f}(t-\tau) + f^{(0)}(t-\tau)] . \end{aligned} \quad (28)$$

The left hand side of (27) represents the total time derivative

$$\frac{d}{dt} \equiv \partial_t + \hat{L} \quad (29)$$

along the trajectory of the evolution of $f'(t, \underline{x}, \underline{v})$ in the

phase space, and the right hand side is a driving field. The

solution (28) is the time integration of the driving field in

the Lagrangian representation denoted by

$$U(t, t-\tau) E'(t-\tau) \cdot \partial[\dots]_{t-\tau} = E'[t-\tau, \underline{x}(t-\tau)] \cdot \partial[\dots]_{t-\tau, \underline{x}(t-\tau), \underline{V}(t-\tau)} \quad (30)$$

Here $\underline{x}(t-\tau)$ is the position along the trajectory at the time $t-\tau$, when it is known that at time t , the position is \underline{x} and the velocity is \underline{v} . The trajectory is governed by:

$$\frac{d\underline{x}(t_1)}{dt_1} = \underline{V}(t_1), \quad \frac{d\underline{V}(t_1)}{dt_1} = \underline{E}(t_1), \quad (31a)$$

where $\underline{V}(t_1)$ is an intermediate variable in

$$\frac{d^2 \underline{x}(t_1)}{dt_1^2} = \underline{E}(t_1). \quad (31b)$$

The dynamical equations (31) governing the trajectory are called

the micro-dynamical equations of transition. The operator

is called the propagator, or "Lagrangian operator". The last

term of (27) does not contribute to the solution, by $A' A_0 = 0$.

The stress in the phase space is thus found to be

$$A^{(0)} \underline{E}' \underline{f}' = - \underline{D}' \cdot \left\{ \partial(\underline{f} + \underline{f}^{(0)}) \right\}. \quad (32)$$

where

$$\underline{D}' = A^{(0)} \int_0^t d\tau \underline{E}'(\underline{t}, \underline{x}) \underline{U}(\underline{t}, \underline{t}-\tau) \underline{E}'(\underline{t}-\tau) \quad (33)$$

is the eddy diffusivity. Hence, upon substituting the stress

into the equation (26a) of macro-evolution, we obtain:

$$A^{(0)} (\partial_t + \hat{L}) \hat{f}_0(\underline{t}, \underline{x}, \underline{v}) = \underline{\partial} \cdot \underline{D}' \left\{ \underline{\partial} (\bar{f} + f^{(0)})_{\underline{t}-\tau} \right\} . \quad (34)$$

In the evolution of the macro-distribution (34) of scale t , we have

$$t > \tau_c^{(0)} \quad (35)$$

and a fortiori

$$t \gg \tau_c' \quad (36)$$

from §1), so that \underline{D}' from (33) becomes a large-time diffusivity

$$\underline{D}' = A^{(0)} \int_0^{t=\infty} d\tau \underline{E}'(\underline{t}, \underline{x}) \underline{U}(\underline{t}, \underline{t}-\tau) \underline{E}'(\underline{t}-\tau) , \quad (37)$$

that asymptotically will take a deterministic form:

$$\underline{D}' = \int_0^\infty d\tau \left\langle \underline{E}'(\underline{t}, \underline{x}) \underline{U}(\underline{t}, \underline{t}-\tau) \underline{E}'(\underline{t}-\tau) \right\rangle , \quad (38)$$

on account of the small correlation time (36) and the deterministic character (22a). Under this circumstance, we reduce (34) to the following kinetic equation

$$(\partial_t + A^{(0)} \hat{L}) f^{(0)} = - \underline{E}^{(0)} \cdot \partial \underline{f} + \partial \cdot \underline{D}' \{ \partial f^{(0)}(t-\tau) \} . \quad (39)$$

It has the form of a Fokker-Planck equation, containing a memory as described by the diffusivity operator $\underline{D}' \{ \}$. By the group-coupling and the scaling procedure, the kinetic equation (39) is explicit in $f^{(0)}$, avoiding the BBGK hierarchy.

KINETIC THEORY OF TRANSITION

The transport theory for the determination of the diffusivity (38) requires the examination of the Lagrangian correlation

$$\begin{aligned} \langle \underline{E}'(t, \underline{x}) U(t, t-\tau) \underline{E}'(t-\tau) \rangle &= \langle \underline{E}'(t, \underline{x}) \underline{E}'(t-\tau, \underline{x}(t-\tau)) \rangle \\ &= \int d\underline{x}' \langle \underline{E}'(t, \underline{x}) \underline{E}'(t-\tau, \underline{x}') \delta[\underline{x}' - \underline{x}(t-\tau)] \rangle . \end{aligned} \quad (40)$$

The trajectory in the phase space

$$\underline{x}(t-\tau) = \underline{x} - \underline{v}\tau + \underline{\hat{\ell}}(-\tau) \quad (41)$$

contains a fluctuating path $\underline{\hat{\ell}}(-\tau)$ in the retrograde transition, i.e. in a time interval $-\tau$, and is self-consistent with the same micro-dynamical state (31) as that of the kinetic equation of velocity distribution (39).

The fluctuating path $\underline{\hat{\ell}}(-\tau)$ is a random function, for which we will develop a kinetic theory of transition.

The first of the micro-dynamical equation of transition (31) can be written in the form

$$\left[\frac{\partial}{\partial \tau} - \underline{\hat{V}}(-\tau) \cdot \frac{\partial}{\partial \underline{\ell}} \right] \hat{P}(-\tau, \underline{\ell}) = 0, \quad (42)$$

called the "master equation of transition", where the velocity

$\underline{\hat{V}}(-\tau)$ has been defined by (31), and

$$\hat{P}(-\tau, \underline{\ell}) \equiv \delta[\underline{\ell} - \underline{\hat{\ell}}(-\tau)] \quad (43)$$

is a probability of transition in the microscopic description.

It is a function of the random variable \underline{l} , and has the

moments:

$$\int d\underline{l} \hat{P} = 1, \quad \int d\underline{l} \underline{l} \hat{P} = \underline{\hat{l}}(-\tau). \quad (44)$$

It is not difficult to verify, by means of the moments (44),

that the master equation (42) will indeed reproduce the micro-dynamical equations (31).

We apply the scalings \bar{A} , A_0 , A' to the master equation (42) and derive the following kinetic equations of transition

$$\frac{\partial \bar{P}(-\tau, \underline{l})}{\partial \tau} = \bar{u} \cdot \frac{\partial \bar{P}}{\partial \underline{l}} + K_{\underline{l}} \left\{ \frac{\partial^2 \bar{P}}{\partial \underline{l} \partial \underline{l}} \right\} \quad (45)$$

$$\frac{\partial P_0(-\tau, \underline{l})}{\partial \tau} = V_0(-\tau) \cdot \frac{\partial P_0}{\partial \underline{l}} + K'_{\underline{l}} \left\{ \frac{\partial^2 P_0}{\partial \underline{l} \partial \underline{l}} \right\}, \quad (46)$$

and the probability of transition in the micro-scale

$$P'(-\tau, \underline{l}) = -\underline{l}'(-\tau) \cdot \frac{\partial P_0(-\tau, \underline{l})}{\partial \underline{l}}, \quad (47)$$

where $\underline{\bar{u}}$ is a drift velocity. The details of derivation, which are similar to those used in the derivation of the kinetic equation of velocity distribution, will not be repeated here.

The coefficient of diffusion, as defined by

$$K_{\underline{\lambda}} = \frac{1}{2} \frac{d}{d\tau} \langle \underline{\tilde{\ell}}(-\tau) \underline{\tilde{\ell}}(-\tau) \rangle \quad (48)$$

is calculated from the micro-dynamics (31) of the path fluctuation, written in the new form :

$$\underline{\tilde{\ell}}(-\tau) = - \int_0^\tau d\tau' \underline{\tilde{V}}(-\tau'), \quad \underline{\tilde{V}}(-\tau) = - \int_0^\tau d\tau' \underline{\tilde{E}}(\tau-\tau'). \quad (49)$$

We find the dispersion

$$\langle \underline{\tilde{\ell}}(-\tau) \underline{\tilde{\ell}}(-\tau) \rangle = 2 \int_0^\tau d\tau' \left(\frac{1}{3} \tau^3 - \frac{1}{2} \tau^2 \tau' + \frac{1}{6} \tau'^3 \right) \langle \underline{\tilde{E}}(\tau) \underline{\tilde{E}}(\tau-\tau') \rangle \quad (50a)$$

and the diffusion coefficient

$$K_{\underline{\lambda}}(\tau) = \tau \int_0^\tau d\tau' (\tau-\tau') \langle \underline{\tilde{E}}(\tau) \underline{\tilde{E}}(\tau-\tau') \rangle. \quad (50b)$$

By scalings, we have:

$$\langle \underline{l}^{(0)}(-\tau) \underline{l}^{(0)}(-\tau) \rangle \approx \frac{1}{4} \tau^4 \langle \underline{E}^{(0)}(\tau) \underline{E}^{(0)}(\tau) \rangle \quad (51a)$$

$$\underline{K}_{\underline{l}}^{(0)}(\tau) \approx \frac{1}{2} \tau^3 \langle \underline{E}^{(0)}(\tau) \underline{E}^{(0)}(\tau) \rangle$$

$$\langle \underline{l}'(-\tau) \underline{l}'(-\tau) \rangle \approx \frac{2}{3} \tau^3 \underline{D}'$$

$$\underline{K}'_{\underline{l}} \approx \tau^2 \underline{D}' \quad (51b)$$

The kinetic equation of transition, in the non-Markoff process of scales (45) and (46), is thus derived from the master equation (42) in self-consistency with the originally stated micro-dynamics of turbulence, in the form (1), (31), or (49), and therefore should not be confounded with the commonly known transition equation which was derived phenomenologically in the hypotheses of a Markoff process.^{13,14} Note also that the kinetic equation of transition determines a retrograde transition of path $\hat{\underline{l}}$.

VI. TRANSPORT THEORY OF DIFFUSIVITY

By the Fourier decomposition, we transform the formula

(38) of diffusivity into

$$D' = \int_0^\infty d\tau \int d\vec{k}' \left\langle \underline{\sigma}'[\vec{k}', \hat{\underline{l}}(-\tau)] \right\rangle e^{i\vec{k}' \cdot \underline{v} \tau}, \quad (52)$$

with the use of (40) and (41). The details of the transform-

ation are elementary and have been omitted. For the sake

of abbreviation, we have introduced the Fourier function

$$\underline{\sigma}'[\vec{k}', \hat{\underline{l}}(-\tau)] \equiv \underline{\sigma}'(\vec{k}') e^{-i\vec{k}' \cdot \underline{\hat{l}}(-\tau)} \quad (53a)$$

and the fluctuating intensity

$$\underline{E}'(\underline{t}, \underline{x}) \underline{E}'(\underline{t}, \underline{x}) \equiv \int d\vec{k}' \underline{\sigma}'(\vec{k}') \quad (53b)$$

of spectral tensor

$$\langle \underline{\sigma}'(\vec{k}') \rangle = \chi \langle \underline{E}'(\vec{k}') \underline{E}'(-\vec{k}') \rangle. \quad (53c)$$

When the Fourier transform is truncated within a length inter-

val 2χ , the truncation factor is $\chi = (\pi/\chi)^d$, where d is

the number of dimensions.

It is to be noted that the integral

$$\int d\underline{k} \langle \sigma'[\underline{k}, \hat{\underline{\ell}}(-\tau)] \rangle e^{i\underline{k} \cdot \underline{v} \tau}, \quad (54a)$$

with a perturbed path $\hat{\underline{\ell}}(-\tau)$, constitutes a Lagrangian corre-

lation of the \underline{E}' - field fluctuations, while the integral

$$\int d\underline{k}' \langle \sigma'[\underline{k}', \hat{\underline{\ell}}(-\tau)] \rangle_{\hat{\underline{\ell}}(-\tau)=0} e^{i\underline{k}' \cdot \underline{v} \tau} \quad (54b)$$

without path perturbations is the Eulerian correlation. Hence

the distinction and the relation between the two correlations

reside in the path perturbations in the Fourier function

$$\langle \sigma' \rangle \equiv \langle \sigma'(\underline{k}') e^{-i\underline{k}' \cdot \hat{\underline{\ell}}(-\tau)} \rangle. \quad (55a)$$

This function can be decomposed into two parts as

$$\begin{aligned} \langle \sigma' \rangle &= \langle \sigma'(\underline{k}') (A_0 + A') e^{-i\underline{k}' \cdot \hat{\underline{\ell}}(-\tau)} \rangle \\ &\equiv \langle \sigma_I' \rangle + \langle \sigma_{II}' \rangle. \end{aligned} \quad (55b)$$

since $A_0 + A'$, with

$$\langle \sigma_{\underline{I}}' \rangle \equiv \langle \underline{s}'(\underline{k}') \rangle \langle A_0 e^{-i\underline{k}' \cdot \hat{\underline{l}}(-\tau)} \rangle \quad (56a)$$

$$\langle \sigma_{\underline{II}}' \rangle \equiv \langle \underline{s}'(\underline{k}') A' e^{-i\underline{k}' \cdot \hat{\underline{l}}(-\tau)} \rangle. \quad (56b)$$

For the analysis of the first part (56a), we calculate:

$$\begin{aligned} A_0 e^{-i\underline{k}' \cdot \hat{\underline{l}}(-\tau)} &\equiv \int d\underline{l} e^{-i\underline{k}' \cdot \underline{l}} P_0(-\tau, \underline{l}) = e^{-i\underline{k}' \cdot \underline{l}_0(-\tau)} \\ A' e^{-i\underline{k}' \cdot \hat{\underline{l}}(-\tau)} &\equiv \int d\underline{l} e^{-i\underline{k}' \cdot \underline{l}} P'(-\tau, \underline{l}) = -i\underline{k}' \cdot \underline{l}'(-\tau) e^{-i\underline{k}' \cdot \underline{l}_0(-\tau)} h' \\ \bar{A} e^{-i\underline{k}' \cdot \underline{l}_0(-\tau)} &\equiv \int d\underline{l} e^{-i\underline{k}' \cdot \underline{l}} \bar{P}(-\tau, \underline{l}) = h_0 h_{\bar{u}}, \end{aligned} \quad (57)$$

from the Fourier transformation of the kinetic equations of

transition (45) - (47), so that we find

$$\langle \sigma_{\underline{I}}' \rangle = \langle \underline{s}'(\underline{k}') \rangle H_D, \quad (58)$$

with the notations:

$$\begin{aligned} H_D &= h_{\bar{u}} h_0 h' \\ h_0 &\equiv \exp \left\{ -\underline{k}' \cdot \underline{k}' : \int_0^\tau d\tau' [K_{\underline{l}}(\tau')] \right\}_0 \\ h' &\equiv \exp \left[-\underline{k}' \cdot \underline{k}' : \int_0^\tau d\tau' K_{\underline{l}}'(\tau') \right] \\ h_{\bar{u}} &= e^{i\underline{k}' \cdot \bar{u} \tau} \end{aligned} \quad (59)$$

The second part (56b) contains the fluctuations already

found in (57), and can thus be written in the form:

$$\begin{aligned}\langle \sigma'_{\Pi} \rangle_j &= -i \underline{k}' \cdot \langle \underline{s}'_j(\underline{k}') \underline{l}'(-\tau) e^{-i \underline{k}' \cdot \underline{l}'_0(-\tau)} \rangle \underline{l}' \\ &= -i \underline{k}' \cdot \langle \underline{s}'_j(\underline{k}') \underline{l}'(-\tau) \rangle H_D.\end{aligned}\quad (60)$$

The degenerated function

$$\langle \underline{s}'_j(\underline{k}') \underline{l}'(-\tau) \rangle = -i \underline{k}' \cdot \langle \underline{l}'(-\tau) \underline{l}'_s(-\tau) \rangle \langle \underline{s}'_j(\underline{k}') \rangle \quad (61)$$

can be considered as a time-integrated flux, so that, by com-

binning (60) and (61), we obtain

$$\langle \sigma'_{\Pi} \rangle = -\xi' \langle \underline{s}'(\underline{k}') \rangle H_D, \quad (62)$$

with

$$\xi' \equiv \underline{k}' \cdot \underline{k}': \langle \underline{l}'(-\tau) \underline{l}'_s(-\tau) \rangle = \frac{2}{3} \tau^3 \underline{k}' \cdot \underline{k}': D_s' \quad (63)$$

in the transport of s' of a path $\underline{l}'_s(-\tau)$ and a diffusivity D_s' .

The results (58) and (62) for the two parts of the Fourier

function (55b) add to

$$\langle \sigma'_{\Pi}[\underline{k}', \underline{l}'(-\tau)] \rangle = (1 - \xi') \langle \underline{s}'(\underline{k}') \rangle H_D. \quad (64)$$

and yield a diffusivity which can be written in the form:

$$\underline{D}' = \int d\underline{k}' \langle \underline{J}'(\underline{k}') \rangle G_D(\underline{k}', \nu). \quad (65)$$

This expresses that the diffusivity is endowed by the spectral tensor $\langle \underline{J}'(\underline{k}') \rangle$ and approaches its equilibrium at a relaxation time

$$G_D(\underline{k}', \nu) \equiv \int_0^\infty d\tau e^{i\underline{k}' \cdot \underline{v} \tau} [1 - \xi'(\underline{k}')] H_D(-\tau, \underline{k}'). \quad (66)$$

where H_D and ξ' have been defined by (59) and (63), respectively.

Note that, since the relaxation time (66) contains \underline{D}' in the integrand, through ξ' and H_D from (63), (59) and (51b), the form (65) is the integral equation for the determination of \underline{D}' .

We conclude that the Lagrangian correlation of the \underline{E}' -field fluctuations is expressed in terms of the spectral

function by the following formula:

$$\langle \underline{E}'(\underline{t}, \underline{x}) U(\underline{t}, \underline{t}-\underline{\tau}) \underline{E}'(\underline{t}-\underline{\tau}) \rangle = \int d\underline{k}' \chi \langle \underline{E}(\underline{k}') \underline{E}'(-\underline{k}') \rangle M(\underline{\tau}, \underline{k}', \underline{v}), \quad (67a)$$

with

$$M(\underline{\tau}, \underline{k}', \underline{v}) = e^{i \underline{k}' \cdot \underline{v} \underline{\tau}} [1 - \xi(\underline{k}')] H_D(-\underline{\tau}, \underline{k}'), \quad (67b)$$

from (64) - (66).

VII. KINETIC BASIS OF THE CASCADE TRANSFER

We write the kinetic equation of velocity distribution

(34) in the following explicit form:

$$\begin{aligned} \partial_t f^{(0)} + A^{(0)} \underline{v} \cdot \underline{\nabla} \hat{f}_0 - \nu \nabla^2 f^{(0)} + A^{(0)} \hat{\underline{E}}_0 \cdot \underline{\partial} f^{(0)} + \underline{E}^{(0)} \cdot \underline{\partial} \bar{f} \\ = \underline{\partial} \cdot \underline{D}' \left\{ \underline{\partial} f^{(0)}(\underline{t}-\underline{\tau}) \right\}. \end{aligned} \quad (68)$$

We take the moment of (68). The zeroth moment gives the

equation of continuity

$$\underline{\nabla} \cdot \underline{u}^{(0)} = 0, \quad (69)$$

and the first moment gives the equation of momentum in the form:

$$\partial_t u_i^{(0)} + \nabla_j (\bar{u}_i u_j^{(0)} + u_i^{(0)} \bar{u}_j + A^{(0)} u_i^{(0)} u_j^{(0)}) - \nu \nabla^2 u_i^{(0)} - E_i^{(0)} = J_i^{(0)} \quad (70)$$

with a collisionless interaction

$$J_i^{(0)} \equiv \int dV v_i \partial \cdot D' \left\{ \partial f^{(0)}(t-\tau) \right\} \quad (71a)$$

The advection $\nabla_j (\bar{u}_i u_j^{(0)})$, the production $\nabla_j (\bar{u}_i u_j^{(0)})$, the inertia $A^{(0)} \nabla_j (u_i^{(0)} u_j^{(0)})$, the viscous damping $\nu \nabla^2 u_i^{(0)}$ and the driving field $E_i^{(0)}$ are all the regular terms recognizable by isolating the macro-group from the hydrodynamical equation (1a) which describes the micro-dynamical state of turbulence. The interaction $J_i^{(0)}$ is new, and can be verified to vanish if the memory should be absent. Since $J_i^{(0)}$ can be identified as a scaled Reynolds stress from the micro-flux of momentum, i.e.

$$J_i^{(0)} = - \nabla_j (A^{(0)} u_i' u_j'), \quad (71b)$$

the memory is indeed essential to turbulence.

From the mixing-length hypothesis, the momentum flux of the gradient type, i.e.

$$A^{(0)} u'_i u'_j = - K'_{j\lambda} \frac{\partial u^{(0)}_i}{\partial x_\lambda} \quad (72)$$

has been established, giving an interaction in the form:

$$J^{(0)}_i = \frac{\partial}{\partial x_\lambda} K'_{j\lambda} \frac{\partial u^{(0)}_i}{\partial x_\lambda} \quad (73)$$

The transport coefficient K' is called the eddy viscosity.

The comparison between the above two forms of interaction suggests a means of isolating this eddy viscosity, provided the transport of the gradient type (72) is valid. On the contrary, when the gradient transport is not valid, as is the case with certain stratified media, the kinetic interaction (70) could determine the correct type of transport appropriate to the problem.

Upon multiplying the momentum equation (70) by $u_i^{(0)}$

and averaging, we obtain the energy equation:

$$\frac{1}{2} \frac{\partial}{\partial t} \langle u^{(0)2} \rangle = P^{(0)} + C^{(0)} - T^{(0)} - \varepsilon^{(0)} - I^{(0)}, \quad (74)$$

describing the rate of change of the kinetic energy, as governed

by the following transport functions:

| | | |
|---------------------|---|------|
| production | $P^{(0)} \equiv - \langle u_n^{(0)} u_i^{(0)} \rangle \nabla_n \bar{u}_i$ | |
| coupling | $C^{(0)} \equiv \langle \underline{u}^{(0)} \cdot \underline{E}^{(0)} \rangle$ | (75) |
| cascade transfer | $T^{(0)} \equiv - \langle \underline{u}^{(0)} \cdot \underline{J}^{(0)} \rangle$ | |
| viscous dissipation | $\varepsilon^{(0)} \equiv \nu \langle (\nabla \cdot \underline{u}^{(0)})^2 \rangle$ | |
| flux transport | $I^{(0)} \equiv - \frac{1}{2} \nabla_n \langle u_n^{(0)} u_i^{(0)2} \rangle$ | |

Among these transport functions, only the mode-coupling

function, or the cascade transfer function

$$\begin{aligned} T^{(0)} &\equiv - \langle u_i^{(0)} J_i^{(0)} \rangle \\ &= - \int dV \, \underline{v}_i \cdot \underline{D} \left\{ \underline{v}_i \cdot \partial \langle u_i^{(0)}(t, \underline{x}) f^{(0)}(t-\tau) \rangle \right\} \end{aligned} \quad (76)$$

possesses a structure that is characteristically kinetic, as involving the correlation

$$\langle u_i^{(0)}(t, \underline{x}) f^{(0)}(t-\tau) \rangle \quad (77)$$

between the fluid velocity $u_i^{(0)}$ and the singlet distribution

$f^{(0)}$. This coupling is analogous to the fluid-particle

interaction which was considered to be the origin of the non-linear Landau damping in plasma turbulence.

In (67) we have derived a formula for the conversion of a Lagrangian correlation into an Eulerian correlation (or its Fourier transform). We apply the same rule for the Lagrangian correlation (77), and obtain:

$$\langle u_i^{(0)}(t, \underline{x}) f^{(0)}(t-\tau) \rangle = \int d\underline{k}'' \chi \langle u_i^{(0)}(\underline{k}'') f^{(0)}(-\underline{k}'', \nu) \rangle M(\tau, \underline{k}'', \nu), \quad (78)$$

where M is defined by (67b).

Now when this Lagrangian correlation is substituted into

(76), we find the transfer function in the form:

$$T^{(0)} = - \iint d\underline{k}'' d\underline{v} \underline{v}_i \partial_{\underline{v}_i} \cdot \underline{D} \left\{ \partial_{\underline{k}'} \chi \langle u_i^{(0)}(\underline{k}'') f^{(0)}(-\underline{k}', \underline{v}) \rangle M(\tau, \underline{k}', \underline{v}) \right\}. \quad (79)$$

Noting that $\underline{D}\{\}$ is an operator of integrations with respect to \underline{k}' and τ , as prescribed by (65) and (66), respectively, we can rewrite (79) in the explicit form, as

follows:

$$T^{(0)} = - \iint d\underline{k}' d\underline{k}'' \int_0^\infty d\tau \int d\underline{v} \underline{v}_i \times \partial_{\underline{v}_i} \left[M(\tau, \underline{k}', \underline{v}) \partial_{\underline{k}'} \chi \langle u_i^{(0)}(\underline{k}'') f^{(0)}(-\underline{k}', \underline{v}) \rangle M(\tau, \underline{k}'', \underline{v}) \right]. \quad (80)$$

VIII. CONCLUSIONS

By means of a scaling procedure, we have developed a kinetic equation of turbulence and a transport theory

of eddy diffusivity. The transport theory is based upon the transformation of a Lagrangian correlation into an

Eulerian correlation by means of transition probabilities

which we have found from a kinetic theory of transition.

The transfer function describes the mode-coupling, or the cascade transfer across the spectrum. It appears in the form of a fluid-particle interaction, and is therefore a kinetic phenomenon like the Landau damping. A memory should exist in this interaction and is generated by the Lagrangian correlation between the fluid velocity and the "particle" velocity distribution. However, a memory-loss should be present for a collisionless dissipation to yield a finite relaxation time. A preliminary examination of the expression of the cascade transfer reveals that the cascade process can persist in the direction of high wavenumbers, so that either a transport along the gradient, or a collisionless dissipation in the explicit

form of a Landau damping can appear. With this distinction, an eddy viscosity or a rate coefficient of Landau damping can be separately analyzed. In other circumstances, e.g. in the atmospheric turbulence with a stable stratification, a reverse cascade may appear, involving a transport counter to the gradient. The reversal is associated with the "gap" phenomenon. ¹⁵

An analytical expression of the cascade transfer of this kind will clarify the gap phenomenon and the concept of "negative viscosity".

ACKNOWLEDGEMENT

This work is performed under the grant NAS8-34622 from the National Aeronautics and Space Administration. The hospitality of the Risø National Laboratory, Roskilde, Denmark, is deeply appreciated.

REFERENCES

1. A. S. Monin, Dokl. Akad. Nauk SSSR 177, 1036 (1967)
Sov. Phys. Doklady 12, 1113 (1968)
2. A. S. Monin, Priklad. Mat. Mekh. 31, 1057 (1967)
Appl. Math. and Mech. 31, 1057 (1967)
3. T. S. Lundgren, Phys. Fluids 10, 969 (1967)
4. T. S. Lundgren, Phys. Fluids 12, 485 (1969)
5. F. R. Ulinich and B. Ya. Lyubimov, Zh. Eksp. Teor. Fiz. 55, 951 (1968) Sov. Phys. - JETP 28, 494 (1969)
6. V. M. Ievlev, Mezhd. Zhid. Gaza, No. 1 (1970).
7. V. M. Ievlev, Dokl. Akad. Nauk SSSR 208, 1044 (1973)
Sov. Phys. Doklady 18, 117 (1973)
8. C. M. Tchen, Phys. Rev., 113, 394 (1959).
9. T. S. Lundgren and Y. B. Pointin, Phys. Fluids 19, 355 (1976).
10. J. Weinstock, Phys. Fluids 21, 887 (1978).
11. J. Weinstock, Phys. Fluids 19, 1702 (1976).
12. C. M. Tchen, Comptes Rendus Acad. Sci. Paris 286A, 605 (1978), also 287B, 175 (1978).
13. C. M. Tchen, Versl. Ned. Akad. Wetensch., Afd. Naturkunde, Amsterdam, 53, 400 (1944).
14. C. M. Tchen, "Mean value and correlation problems connected with the motion of small particles suspended in turbulent fluid," 125 p. Martinus Nijhoff Publisher, The Hague, 1947.
15. J. Weinstock, J. Atmos. Sci. 37, 1542 (1980).

CHAPTER B: QUARTER TWO

Kinetic Theory of Turbulent Transfer with Double Memory-Loss

C.M. Tchen

The Graduate Center and The City College of
The City University of New York, N.Y. 10031

Abstract

The transfer function governs the mode-coupling in strong turbulence. It is investigated here by means of a kinetic equation of turbulence, in consistency with the hydrodynamical system that describes the micro-dynamical state of turbulence. The equation of spectral balance is obtained by taking the moment of the scaled kinetic equation for the one-point distribution without the need of the two-point distribution. It is found that the transfer function is governed by two Lagrangian correlation functions and therefore by two memory-loss functions. The first correlation relates to two field fluctuations and finds the diffusivity as an integral operator. This perpetuates the memory to the second correlation which contains its own memory from the fluctuations of velocity and distribution of velocities. This correlation is known as the "wave-particle interaction" in plasmas. The competition between the two memory-loss functions and the cut-off of the memory form the essential basis for the closure of turbulence. We find two forms of transfer functions, for small and large scales, respectively. The

first form belongs to the gradient transport and isolates an eddy viscosity. The second form belongs to a collisionless damping without gradient, and isolates a rate coefficient of collisionless damping. The transfer function for the large wavenumbers is applied to the derivation of the Kolmogoroff law of turbulence with the analytical determination of the numerical coefficient.

1. INTRODUCTION

The micro-dynamical state of fluid turbulence is usually described by a hydrodynamical system, for example the Navier-Stokes equation of motion. This equation is nonlinear in the velocity function and is inhomogeneous by having a driving field which may be the pressure gradient. On the other hand, in the kinetic representation the velocity is an independent variable and does not cause a nonlinearity, while the driving field forms a part of the advection and does not warrant an inhomogeneous equation. It is known that a homogeneous equation for the micro-dynamical state of turbulence can more explicitly give an analytical specification of the path perturbations, since the homogeneous equation contains a detailed differential operator of perturbations and therefore will yield by inversion an exact evolution operator.

Unlike the kinetic theory of gases, the kinetic theory of the spectral structure of turbulence usually requires a

two-point distribution function, since the spectral function is the Fourier transform of the two-point correlation function of velocity fluctuations. We see that such a procedure would involve a hierarchy in the form of the Bogoliubov hierarchy¹⁻⁷. In order to confine ourselves to a single kinetic equation of one-point distribution and thus avoid the Bogoliubov hierarchy, we introduce a procedure of scaling into a mean group and groups of macro- and micro-fluctuations, representing three processes of transport: macro-evolution, micro-transport property, and relaxation^{8,9}. These groups are in their increasing order of incoherence. The relaxation process, as the most random group, provides a memory-loss by path perturbations. Thus the closure lies in the memory-loss that is necessary for the transport property to reach its equilibrium, rather than in closing the Reynolds stresses or other higher order transport functions.

Our first task is to derive a macro-kinetic equation, not through the closure of the Bogoliubov hierarchy, but by using a scaling procedure. The result is a macro-kinetic equation with a diffusivity serving as an integral operator, so that the memory can be extended to the distribution function that follows. Consequently, the kinetic equation takes the form of a mixture of the Fokker-Planck differential equation and the Boltzmann integral equation (Section II).

The kinetic equation of one-point macro-distribution can be transformed into a continuum representation by means of the moment method, and subsequently, upon multiplication by a macro-velocity, can derive an energy equation for the spectral balance of turbulence, bypassing the need for the two-point distribution function. Since the diffusivity acts as an operator, the distribution function that follows takes a Lagrangian form, and, together with the macro-velocity, will form a Lagrangian correlation. In this way, a transfer function is obtained, describing the mode-coupling across the spectrum, and is founded on two Lagrangian correlations: one from the field fluctuations for forming the diffusivity, and the other from the macro-distribution for describing the "particle-fluid" interaction. Since each Lagrangian correlation presents a memory loss, the transfer function possesses two memory-loss functions, the analysis of which must be based on the Lagrangian-Eulerian transformation. The said transformation is in the phase-space and differs from an analogous transformation¹⁰⁻¹² for diffusion in the configuration space (Section III). The competitive interplay of the two memory-loss functions provides us with a means of closure for the derivation of the transfer function in the form of a gradient transport or in the form of a collisionless damping without a macroscopic gradient (Section IV). In this manner, we are able

to isolate an eddy viscosity and a rate of collisionless damping (Section V).

Finally, we test the applicability of our kinetic method to the inertia turbulence. We derive analytically the Kolmogoroff law^{13,14} and its numerical coefficient (Section VI).

II. THE SCALED KINETIC EQUATION OF TURBULENCE

The hydrodynamical equations

$$\left(\partial_t + \hat{\mathbf{u}} \cdot \nabla - \nu \nabla^2 \right) \hat{\mathbf{u}} = \hat{\mathbf{E}}, \quad \nabla \cdot \hat{\mathbf{u}} = 0, \quad (1)$$

which describe the micro-dynamical state of turbulence, govern the total velocity $\hat{\mathbf{u}}$ as driven by a field $\hat{\mathbf{E}}$. This field may consist of the pressure gradient, the buoyancy, or any other fluctuating forces, including the random noise.

By the Reynolds decomposition, we can separate the total field

$$\hat{\mathbf{E}} = \bar{\mathbf{E}} + \mathbf{E} \quad (2a)$$

into a mean field $\langle \hat{\mathbf{E}} \rangle = \bar{\mathbf{E}}$ and a fluctuation \mathbf{E} . In an analogous way, we can decompose the field fluctuation

$$\mathbf{E} = \mathbf{E}^{(0)} + \mathbf{E}' \quad (2b)$$

into a macro-field $\bar{E}^{(o)}$ and a micro-field \tilde{E}' , with unequal correlation times

$$\tau_c^{(o)} > \tau_c' , \quad (3)$$

in the increasing order of incoherence. The scaling can be made by means of the operators

$$\bar{A} \equiv \langle \dots \rangle, A^{(o)}, A' , \quad (4a)$$

which select the deterministic "global average", the macro- and the micro-fluctuations, respectively. The macro-fluctuation evolves in a medium possessing a transport property that is carried by the micro-field fluctuations. Finally, a relaxation is needed for the transport property to approach its equilibrium. This is made possible by a memory-loss. Thus the three transport processes of evolution, transport property, and relaxation form our main framework for the closure of turbulence. For the sake of convenience, we introduce an operator

$$A_o = \bar{A} + A^{(o)} , \quad (4b)$$

to represent an "accumalted macro-group" of fluctuating field

$$\tilde{E}_o = \bar{\tilde{E}} + \tilde{E}^{(o)} , \quad (4c)$$

and path

$$\tilde{l}_0 = \bar{l} + \tilde{l}^{(0)} . \quad (4d)$$

It is to be noted that the averages by \bar{A} , A_0 are ensemble averages as specified by appropriate distribution functions. The groups as scaled by \bar{A} , $A^{(0)}$, A' , only distinguish themselves by their degree of incoherence, and therefore may overlap in their spectral distributions with respect to frequency and wavenumber.

The deterministic functions, which result from the global averages of the scaled fluctuating functions, inherit the same "rank" notations

$$(\)^0 , (\)' , (\)_0$$

of the originating scaled fluctuating functions.

The hydrodynamical system of equations describing the micro-dynamical state of turbulence is usually complicated by the velocity function and the driving force which make the equations nonlinear and inhomogeneous. In order to avoid these difficulties, we adopt the kinetic approach. This transforms the velocity function into an independent variable, and incorporates the driving force into the advection in the phase space. This procedure renders the equations homogeneous and eliminates most nonlinearities, except the one representing the coupling between the driving field and the distribution function in the so-called "wave-particle" interaction. This last nonlinearity does not need

our immediate attention. We may postpone after the closure and solve by an equation of state relating the fluctuations of field and velocity. These advantages will provide us with a simpler statistical basis of closure.

As the point of departure, we write the following macrokinetic equation⁸:

$$A^{(0)} \left(\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla} + \hat{E} \cdot \underline{\partial} \right) f^{(0)} = - \underline{E}^{(0)} \cdot \underline{\partial} \bar{f} + \underline{\partial} \cdot \underline{D}' \left\{ \underline{\partial} f^{(0)}(t-\tau) \right\}, \quad (5)$$

where $\bar{f}(t, \underline{x}, \underline{v})$ and $f^{(0)}(t, \underline{x}, \underline{v})$ are the mean and the macro-distributions, respectively, and

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t}, \quad \underline{\nabla} \equiv \frac{\partial}{\partial \underline{x}}, \quad \underline{\partial} \equiv \frac{\partial}{\partial \underline{v}}.$$

The eddy diffusivity is

$$\underline{D}' = \int_0^\infty d\tau \left\langle \underline{E}'(t, \underline{x}) \underline{U}(t, t-\tau) \underline{E}'(t-\tau) \right\rangle, \quad (6a)$$

or, in the simplified form:

$$\underline{D}' = \int_0^\infty d\tau \left\langle \underline{E}'(t) \underline{E}'(t-\tau) \right\rangle, \quad (6b)$$

where the Lagrangian representation of the micro-field is simply written as

$$\begin{aligned} \underline{U}(t, t-\tau) \underline{E}'(t-\tau) &\equiv \underline{E}'(t-\tau) \\ &= \underline{E}'[t-\tau, \underline{x}(t-\tau)], \end{aligned} \quad (6c)$$

and the Eulerian field at time t is written as

$$\tilde{E}(t, \tilde{x}) = \tilde{E}(t)$$

since $\tilde{x}(t) = \tilde{x}$ in $\tilde{E}(t, \tilde{x}(t)) = \tilde{E}(t)$. The propagator, or the Lagrangian evolution operator $U(t, t-\tau)$ describes the evolution of the perturbed trajectory. The diffusivity may serve the role of an integral operator if it is written as $\tilde{D}'\{\}$.

III. THE LAGRANGIAN-EULERIAN TRANSFORMATION IN THE PHASE SPACE.

THE LOSS AND CUTOFF OF THE MEMORY

By means of the Fourier transformation in \tilde{k} -space, the Lagrangian correlation can be written in the following form:

$$\langle \tilde{E}(t, \tilde{x}) \tilde{E}(t-\tau) \rangle = \int d\tilde{k}'' \tilde{h}_{\omega}(-\tau, \tilde{k}'') \tilde{h}_{\omega}(\tau, \tilde{k}'') \langle \tilde{J}'(\tilde{k}'') e^{i\tilde{k}'' \cdot \hat{\tilde{l}}(-\tau)} \rangle. \quad (7)$$

It contains the spectral function

$$\tilde{J}'(\tilde{k}'') = \chi \langle \tilde{E}'(\tilde{k}'') \tilde{E}'(-\tilde{k}'') \rangle$$

giving the mean intensity

$$\int d\tilde{k}'' \langle \tilde{J}'(\tilde{k}'') \rangle = \langle \tilde{E}'(t, \tilde{x}) \tilde{E}'(t, \tilde{x}) \rangle.$$

where

$$\chi = (\pi/\chi)^d$$

is a factor of Fourier truncation within a length interval $2X$ in d dimensions. The Fourier transformation has the Fourier kernels

$$h_{\omega}(-\tau, k'') \equiv e^{i\omega_k \tau}, \quad h_{\mathbf{v}}(-\tau, k'', \mathbf{v}) \equiv e^{-i\mathbf{k}'' \cdot \mathbf{v} \tau}. \quad (8)$$

The frequency $\omega_k = \omega_k(k'')$ is a function of k'' as determined by a certain dispersion relation, for example, reflecting the interaction between internal gravity wave and turbulence¹⁵⁻¹⁷. The perturbed path $\hat{\mathbf{l}}(-\tau)$ forms a fluctuating orbit function $e^{i\mathbf{k}'' \cdot \hat{\mathbf{l}}(-\tau)}$. Since we can decompose the path $\hat{\mathbf{l}}$ into groups

$$\hat{\mathbf{l}} = \bar{\mathbf{l}} + \mathbf{l}, \quad \mathbf{l} = \mathbf{l}^{(0)} + \mathbf{l}', \quad \mathbf{l}^{(0)} = \bar{\mathbf{l}} + \mathbf{l}'^{(0)},$$

we find the corresponding "scaled orbit functions", or briefly, the "orbit components":

$$\bar{h} \equiv e^{i\mathbf{k}'' \cdot \bar{\mathbf{l}}(-\tau)}, \quad h^{(0)} \equiv \langle e^{i\mathbf{k}'' \cdot \mathbf{l}^{(0)}(-\tau)} \rangle, \quad h' \equiv \langle e^{i\mathbf{k}'' \cdot \mathbf{l}'(-\tau)} \rangle. \quad (9)$$

On a separate occasion, we have developed a theory of transition probability for the orbital motion and determined the following correlation function to be¹⁸⁻²⁰:

$$\langle \hat{J}'(\mathbf{k}'') e^{i\mathbf{k}'' \cdot \hat{\mathbf{l}}(-\tau)} \rangle = \langle \hat{J}'(\mathbf{k}'') \rangle \langle e^{i\mathbf{k}'' \cdot \hat{\mathbf{l}}(-\tau)} \rangle h'_{\mathbf{f}}. \quad (10)$$

The first two factors on the right-hand side represent the factorization when the fluctuating functions

$$\underset{\sim}{s}' \quad \text{and} \quad e^{i\underset{\sim}{k} \cdot \underset{\sim}{l}} \quad (11)$$

are mutually independent, and h'_{ξ} is a correction factor when the hypothesis of independence does not hold.

With the substitution of (10) into (6), we find that the Lagrangian correlation

$$W_L'(\tau) \equiv \langle E'(\underset{\sim}{t}_x) E'(\underset{\sim}{t}-\tau) \rangle = \int d\underset{\sim}{k}'' \langle \underset{\sim}{s}'(\underset{\sim}{k}'') \rangle \hat{M}(-\tau, \underset{\sim}{k}'', \underset{\sim}{v}) \quad (12)$$

has, in the Fourier integral, a product of the spectral distribution $\langle \underset{\sim}{s}'(\underset{\sim}{k}'') \rangle$ with the orbit function

$$\hat{M} = h_{\omega} h_v \bar{h} h^{(0)} h' h'_{\xi} \quad (13a)$$

The components may form the groups

$$h_{*v} \equiv h_v \bar{h} \quad \text{and} \quad M \equiv h^{(0)} h'_{*} \quad (13b)$$

representing the streaming by $\underset{\sim}{v}, \bar{\underset{\sim}{u}}, \bar{\underset{\sim}{E}}$ and the memory-loss by the macro- and micro-fluctuations, respectively. For the sake of convenience, we write the micro-component

$$h'_{*} = h' h'_{\xi} \quad (13c)$$

that incorporates the inter-dependence between the fluctuating intensity $\underset{\sim}{s}'$ and the path fluctuations $\underset{\sim}{l}'$; also we write the accumulated macro-component as

$$h_0 \equiv \bar{h} h^{(0)}. \quad (13d)$$

The function M will be called the "memory-loss function".

In particular, if the streaming predominates, we can degenerate (13a) into

$$\hat{M} \cong h_\nu \bar{h}, \text{ with } M \cong 1,$$

so that the Lagrangian correlation (12) is reduced to the Eulerian correlation by the relation:

$$\begin{aligned} W_L'(\tau) &\cong W_E'(\tau) = \left\langle E'(t, x) E'(t-\tau, x-\bar{\ell}) \right\rangle \\ &= \int dk'' \left\langle s(k'') \right\rangle h_\omega(-\tau, k'') h_\nu(-\tau, k'', \nu) \bar{h}(-\tau, k''). \end{aligned} \quad (14)$$

This is the well-known relation under the hypothesis of "frozen turbulence", and is valid in weak turbulence. In the general case where the turbulence is not frozen, we have the relationship

$$W_L'(\tau) = W_E' \{ M(\tau) \} \quad (15)$$

between the two correlations by using $W_E' \{ \}$ as an integral operator. Note that

$$W_L'(\tau) \leq W_E'(\tau)$$

since $M \leq 1$.

In the derivation of a transport property or a transport function in equilibrium upon the τ -integration of a Lagrangian correlation, it is obvious that the memory should not perpetuate indefinitely but should ultimately be cut off, as a requirement of the closure. We see that the memory function M consists of two components: The macro-component $h^{(0)}$ incorporates the strength and the decay of the memory. The micro-component h'_* , as belonging to the most random group at the tail of the memory-chain, has the function of cutting off the memory as the result of the diffusion by the micro-fluctuations.

Recall that the diffusions by \underline{u} - and \underline{E} -fluctuations are governed by the dynamical equations of the orbital motions

$$\frac{d\hat{\underline{l}}}{dt} = \hat{\underline{u}} \quad (16a)$$

and

$$\frac{d\hat{\underline{l}}}{dt} = \hat{\underline{u}} , \quad \frac{d\hat{\underline{u}}}{dt} = \hat{\underline{E}} , \quad (16b)$$

respectively.

We perform the scalings and the ensemble averages by means of the scaled probabilities of transition^{18,19}.

The dynamical system (16b) gives the paths

$$\hat{\underline{l}}(-\tau) = -\hat{\underline{u}}\tau - \frac{1}{2}\hat{\underline{E}}\tau^2, \quad \hat{\underline{l}}^{(0)}(-\tau) = -\frac{1}{2}\hat{\underline{E}}^{(0)}\tau^2, \quad (17a)$$

and a macro-variance

$$\langle \tilde{l}^{(0)}(-\tau) \tilde{l}^{(0)}(-\tau) \rangle = \frac{1}{4} \tau^4 \langle \tilde{E}^{(0)}(\tau) \tilde{E}^{(0)}(\tau) \rangle \quad (17b)$$

belonging to the unmatured diffusion, or small-time diffusion.

The micro-variance can be written in the form

$$\langle \tilde{l}'(-\tau) \tilde{l}'(-\tau) \rangle \underset{\approx}{\cong} 2 \underset{\approx}{K'} \tau, \quad (18a)$$

belonging to the matured diffusion, or large-time diffusion with a transport coefficient

$$\underset{\approx}{K'} = \int_0^{\tau \rightarrow \infty} d\tau' \langle \tilde{u}'(\tau) U(\tau, \tau - \tau') \tilde{u}'(\tau - \tau') \rangle. \quad (18b)$$

Two forms of propagators $U(\tau, \tau - \tau')$ may be visualized, depending on the orbit evolution, whether it be perturbed by the \tilde{u} -fluctuations or by the \tilde{E} -fluctuations, giving an eddy diffusivity or an eddy viscosity, respectively. Alternatively, the micro-variance in the diffusion by \tilde{E} -fluctuations following (16b) is expected to deliver an eddy viscosity too, in an indirect way through $\underset{\approx}{D'}$ and after the closure. For this reason, we shall assume a memory cut-off and a closure by (18a). A rough evaluation of $\underset{\approx}{K'}$ could be made by relating it to the noise spectrum of the Brownian problem, or to the diffusion model (16a). But from a realistic viewpoint, $\underset{\approx}{K'}$ must be related, in a self-consistent

way, to that same eddy viscosity which governs the mode transfer by small scale fluctuations (Section V). We adopt the latter viewpoint.

The variances (17b) and (18a) determine the orbit components:

$$h^{(0)} = e^{-d^{(0)}}, \quad h' = e^{-d'}, \quad (19a)$$

with

$$\begin{aligned} d^{(0)} &\equiv \frac{1}{2} k'' k'' : \langle \tilde{l}^{(0)}(-\tau) \tilde{l}^{(0)}(-\tau) \rangle \\ d' &= \frac{1}{2} k'' k' : \langle \tilde{l}'(-\tau) \tilde{l}'(-\tau) \rangle. \end{aligned} \quad (19b)$$

It is to be mentioned that, unlike h' which depends on the variance $\langle \tilde{l}'(-\tau) \tilde{l}'(-\tau) \rangle$ of one path $\tilde{l}'(-\tau)$, the component

$$h_{\xi}' \equiv 1 - \xi' \quad (20)$$

contains a function ξ' that depends on the micro-fluctuations of two paths, one is $\tilde{l}'(-\tau)$ and the other is $\tilde{l}_s'(-\tau)$ carrying the field intensity s' , so that

$$\xi'(-\tau, k') = \frac{1}{4} \tau^4 k'' k'' : \langle \tilde{E}(0) \tilde{E}(\tau) \rangle. \quad (21)$$

For small τ , the two paths have still to grow, so that ξ' is small. For large τ , the two paths are too far apart and

lose their correlation, so that ξ' tends again to zero. It is for the intermediate τ that ξ' has a finite value, so that (20) may even become negative, yielding a negative portion of the Lagrangian correlation (15).

Now we integrate the Lagrangian correlation with respect to τ , to find the diffusivity in the phase space:

$$\tilde{D}' = \int_0^\infty d\tau \int d\tilde{k}'' \langle \tilde{\rho}(\tilde{k}'') \rangle \hat{M}(-\tau, \tilde{k}', \nu). \quad (22)$$

This expression can also be used as an integral operator for integrations with respect to τ and \tilde{k}'' . The memory-loss function M , as forming a part of \hat{M} , governs the loss of memory for the diffusivity to approach its equilibrium. The time integral

$$\int_0^\infty d\tau \hat{M}(-\tau, \tilde{k}', \nu) \quad (23)$$

will be called the relaxation time.

IV. CASCADE TRANSFER WITH DOUBLE MEMORY-LOSS

We take the moment of the kinetic equation (5) to obtain the following moment equation:

$$\partial_t u_i^{(0)} + \nabla_j (\bar{u}_i u_j^{(0)} + u_i^{(0)} \bar{u}_j + A^{(0)} u_i^{(0)} u_j^{(0)}) - \nu \nabla^2 u_i^{(0)} = J_i^{(0)}, \quad (24)$$

with

$$J_i^{(0)} \equiv \int d\mathbf{v} \, v_i \, \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}' \left\{ \frac{\partial}{\partial \mathbf{v}} f^{(0)}(t-\tau) \right\}. \quad (25)$$

Upon multiplying (24) by $u_i^{(0)}$ and averaging, we obtain the energy equation:

$$\frac{1}{2} \frac{\partial}{\partial t} \langle u^{(0)2} \rangle = P^{(0)} + C^{(0)} - T^{(0)} - \epsilon^{(0)} - I^{(0)}, \quad (26)$$

where the transport functions are as follows:

$$\text{production} \quad P^{(0)} \equiv - \langle u_i^{(0)} u_n^{(0)} \rangle \nabla_n \bar{u}_i$$

$$\text{coupling} \quad C^{(0)} \equiv \langle u^{(0)} \cdot E^{(0)} \rangle$$

$$\text{transfer} \quad T^{(0)} \equiv - \langle u^{(0)} \cdot J^{(0)} \rangle$$

$$\text{dissipation} \quad \epsilon^{(0)} \equiv \chi \langle (\nabla u^{(0)})^2 \rangle$$

$$\begin{array}{l} \text{flux in energy} \\ \text{transport} \end{array} \quad I^{(0)} \equiv - \frac{1}{2} \nabla_n \langle u_n^{(0)} u_i^{(0)2} \rangle. \quad (27)$$

The momentum equation and the energy equation are not purely hydrodynamical, since they contain the distribution function $f^{(0)}$ through $J^{(0)}$ and $T^{(0)}$. The function $J^{(0)}$ represents a collisionless interaction, and the transfer function $T^{(0)}$ represents a transfer between the two portions of the spectrum of macro- and micro-energies, respectively. This transfer originates from the "particle-fluid" inter-

action, as shown by the new Lagrangian correlation

$$\varphi_i^{(0)}(-\tau, \underline{x}, \underline{v}) \equiv \langle u_i^{(0)}(\underline{x}) f^{(0)}(\underline{x}, \underline{v}) \rangle \quad (28)$$

between the fluid velocity $u_i^{(0)}$ and the distribution $f^{(0)}$ of the "particles". Thus the transfer function is more specifically

$$\begin{aligned} T^{(0)} &= - \langle u_i^{(0)} \cdot J^{(0)} \rangle \\ &= - \int d\underline{v} \, v_i \, \partial \cdot \underline{D}' \left\{ \partial \varphi_i^{(0)}(-\tau, \underline{x}, \underline{v}) \right\} \end{aligned} \quad (29)$$

from (25), (27) and (28).

By applying the rule (12), we can transform the new Lagrangian correlation into

$$\varphi_i^{(0)}(-\tau, \underline{x}, \underline{v}) = \int d\underline{k}' \, \varphi_i^{(0)}(-\tau, \underline{k}', \underline{v}) \hat{M}(-\tau, \underline{k}', \underline{v}), \quad (30a)$$

with

$$\varphi_i^{(0)}(\underline{k}', \underline{v}) \equiv \chi \langle u_i^{(0)}(\underline{k}') f^{(0)}(-\underline{k}', \underline{v}) \rangle. \quad (30b)$$

and get the transfer function in the form:

$$T^{(0)} = - \int d\underline{k}' \int d\underline{v} \, v_i \, \partial \cdot \underline{D}' \left\{ \partial \varphi_i^{(0)}(\underline{k}', \underline{v}) \hat{M}(-\tau, \underline{k}', \underline{v}) \right\}. \quad (31)$$

This form can be made more explicit by inserting the expression (22) for \underline{D}' , as follows:

$$T^{(0)} = - \int \int d\tilde{k}' d\tilde{k}'' \langle j_{j\tilde{n}}'(\tilde{k}'') \rangle \int_0^\infty d\tau N_{j\tilde{n}}^{(0)}(-\tau, \tilde{k}', \tilde{k}''). \quad (32)$$

Here we have introduced for the sake of convenience:

$$N_{j\tilde{n}}^{(0)}(-\tau, \tilde{k}', \tilde{k}'') \equiv \int d\tilde{v} \, v_i \, \partial_j \left\{ \hat{M}(-\tau, \tilde{k}'', \tilde{v}) \partial_{\tilde{n}} \left[\hat{M}(-\tau, \tilde{k}', \tilde{v}) \varphi_i^{(0)}(\tilde{k}', \tilde{v}) \right] \right\}, \quad (33a)$$

or, more briefly,

$$N_{j\tilde{n}}^{(0)} \equiv \int d\tilde{v} \, v_i \, \partial_j \left\{ \hat{M}(\tilde{k}'') \partial_{\tilde{n}} \left[\hat{M}(\tilde{k}') \varphi_i^{(0)}(\tilde{k}') \right] \right\}. \quad (33b)$$

where the dependence of $\hat{M}(\tilde{k}'')$, $\hat{M}(\tilde{k}')$ on $-\tau, \tilde{v}$, and of $\varphi_i^{(0)}(\tilde{k}')$ on \tilde{v} may be considered to be understood.

By expanding the multiple derivatives $\partial_j, \partial_{\tilde{n}}$, we have:

$$N_{j\tilde{n}}^{(0)} \equiv \int d\tilde{v} \, v_i \left[\hat{M}(\tilde{k}'') \partial_j \partial_{\tilde{n}} \hat{M}(\tilde{k}') \varphi_i^{(0)}(\tilde{k}') + \hat{M}(\tilde{k}'') \hat{M}(\tilde{k}') \partial_j \partial_{\tilde{n}} \varphi_i^{(0)}(\tilde{k}') + X_i^{(0)}(\tilde{k}', \tilde{k}'') \right], \quad (34)$$

with

$$\begin{aligned} X_i^{(0)} \equiv & \partial_j \hat{M}(\tilde{k}'') \partial_{\tilde{n}} \hat{M}(\tilde{k}') \varphi_i^{(0)}(\tilde{k}') \\ & + \partial_j \hat{M}(\tilde{k}'') \hat{M}(\tilde{k}') \partial_{\tilde{n}} \varphi_i^{(0)}(\tilde{k}') + \hat{M}(\tilde{k}'') \partial_{\tilde{n}} \hat{M}(\tilde{k}') \partial_j \varphi_i^{(0)}(\tilde{k}') \\ & + \hat{M}(\tilde{k}'') \partial_j \hat{M}(\tilde{k}') \partial_{\tilde{n}} \varphi_i^{(0)}(\tilde{k}') \end{aligned} \quad (35)$$

The second term on the right-hand side of (34) gives, after two successive integrations by parts:

$$\begin{aligned}
\int d\underline{v} \, v_i \, \hat{M} \hat{M} \, \partial_j \partial_n \varphi_i^{(0)} &= \int d\underline{v} \, \varphi_i^{(0)} \partial_j \partial_n (v_i \hat{M} \hat{M}) \\
&= \int d\underline{v} \, v_i \, \varphi_i^{(0)} \partial_j \partial_n (\hat{M} \hat{M}) + \int d\underline{v} \, \varphi_i^{(0)} (\delta_{in} \partial_j + \delta_{ij} \partial_n) (\hat{M} \hat{M}), \quad (36)
\end{aligned}$$

with the obvious condition

$$\varphi_i^{(0)}(\underline{k}', \underline{v}) = 0 \quad \text{for} \quad |\underline{v}| = \infty,$$

so that, after a substitution, (34) becomes

$$\begin{aligned}
N_{jr}^{(0)} &= \int d\underline{v} \, v_i \, \varphi_i^{(0)}(\underline{k}') \left[\hat{M}(\underline{k}'') \partial_j \partial_n \hat{M}(\underline{k}') + \partial_j \partial_n (\hat{M}(\underline{k}'') \hat{M}(\underline{k}')) \right] \\
&\quad + \int d\underline{v} \, v_i \, X_{jri}^{(0)}(\underline{k}', \underline{k}'') + \int d\underline{v} \, Y_{jr}^{(0)}(\underline{k}', \underline{k}''), \quad (37)
\end{aligned}$$

with

$$Y_{jr}^{(0)}(\underline{k}', \underline{k}'') = \int d\underline{v} \, \varphi_i^{(0)}(\underline{k}') \left[\delta_{in} \partial_j (\hat{M}(\underline{k}'') \hat{M}(\underline{k}')) + \delta_{ij} \partial_n (\hat{M}(\underline{k}'') \hat{M}(\underline{k}')) \right]. \quad (38)$$

As far as their real values are concerned,

$N_{jr}^{(0)}(\underline{k}', \underline{k}'')$, $\varphi_i^{(0)}(\underline{k}')$, $\hat{M}(\underline{k}'')$ and $\hat{M}(\underline{k}')$ are even functions of \underline{k}'' and \underline{k}' , so that the derivatives

$$\partial \hat{M}(\underline{k}'') \quad \text{and} \quad \partial \hat{M}(\underline{k}')$$

will generate odd functions, and will not contribute to the integrations with respect to \underline{k}'' and \underline{k}' in the infinite domain. Therefore they can be deleted from the expression

of $N_{jr}^{(o)}$ for the purpose of calculating $T^{(o)}$ from (32).

The result

$$N_{jr}^{(o)} = \int d\underline{v} \, v_i \, \varphi_i^{(o)}(\underline{k}') \left[\hat{M}(\underline{k}'') \partial_j \partial_r \hat{M}(\underline{k}') + \partial_j \partial_r (\hat{M}(\underline{k}'') \hat{M}(\underline{k}')) \right], \quad (39)$$

is in the form of the moment of $\varphi_i^{(o)}$ as driven by the differentials $\partial_j \partial_r$ of the orbit functions. Hence the transfer function (32) takes the form

$$T^{(o)} = - \iint d\underline{k}' d\underline{k}'' \langle \mathcal{J}_{jr}(\underline{k}'') \rangle \\ \times \int_0^\infty d\tau \int d\underline{v} \, v_i \, \varphi_i^{(o)}(\underline{k}') \left[\hat{M}(\underline{k}'') \partial_j \partial_r \hat{M}(\underline{k}') + \partial_j \partial_r (\hat{M}(\underline{k}'') \hat{M}(\underline{k}')) \right], \quad (40)$$

and is seen to be controlled by two memory-loss functions $M(\underline{k}'')$ and $M(\underline{k}')$ as imbedded in $\hat{M}(\underline{k}'')$ and $\hat{M}(\underline{k}')$.

V. EDDY VISCOSITY AND COLLISIONLESS DAMPING

It is to be noted that the \underline{v} -dependent orbit functions have their slowly varying component h'_* and their rapidly varying component h_v . Thus the differential $\partial \hat{M}$ will be mainly contributed by the differential ∂h_v of the rapidly varying component in the manner

$$\partial \hat{M} \cong -ik \tau \hat{M}. \quad (41)$$

We notice that the two orbit functions

$$\hat{M}(-\tau, \underline{k}', \underline{v}) \quad \text{and} \quad \hat{M}(-\tau, \underline{k}'', \underline{v})$$

compete in their roles of loss of memory for the approach of the transfer function (40) to equilibrium, under the scaling conditions

$$0 \leq k' \leq k \leq k'' \leq \infty . \quad (42)$$

Thus it suffices to select the function that is most effective in this role. On this basis, we shall calculate the asymptotic expressions by considering small and large scales. Since the path amplitudes $l^{(0)}, l'$ and the transport coefficients $K_{\frac{1}{2}l}^{(0)}, K'_{\frac{1}{2}l}$ are separated by k , the small and large values of k correspond to :

$$l^{(0)} > l' \quad (43a)$$

and

$$l^{(0)} < l' , \quad (43b)$$

respectively.

Case (a). Large k

For large k , or for even larger k'' , such that

$$k'' \gg k' \quad (44)$$

in conjunction with (42) and (43a), we can attribute the dominant role of memory-loss to the function

$$\hat{M}(-\tau, \underline{k}'') \cong h_{\omega}(-\tau, \underline{k}'') h_0(-\tau, \underline{k}'') h'(-\tau, \underline{k}'') . \quad (45)$$

which is independent of \underline{v} , while leaving the role of streaming to the \underline{v} -dependent orbit-function

$$\hat{M}(-\tau, \underline{k}') \cong h_{\omega}(-\tau, \underline{k}') h_v(-\tau, \underline{k}', \underline{v}) . \quad (46)$$

The independence of (45) on \underline{v} is seen from (18) and (19), after having assumed $h'_{\xi} \cong 1$ or $\xi' \cong 0$. This approximation is legitimate, because in the integration (40) with respect to τ , the role of h'_{ξ} in $\hat{M}(-\tau, \underline{k}'', \underline{v})$ is seen from (20) and (21) to be taken over by h' and h_0 for small and large τ , respectively. Hence we reduce (39) and (40) to

$$N_{jn}^{(0)} \Big|_{\text{large } k} = -k_j k'_n \tau^2 h_0(-\tau, \underline{k}'') h'(-\tau, \underline{k}'') \int d\underline{v} v_i \varphi_i^{(0)}(\underline{k}') h_v(-\tau, \underline{k}', \underline{v}) , \quad (47)$$

and

$$T^{(0)} \Big|_{\text{large } k} = \int d\underline{k}'' \langle j'_{jn}(\underline{k}'') \rangle \int d\underline{k}' k_j k'_n \int d\underline{v} v_i \varphi_i^{(0)}(\underline{k}', \underline{v}) G(\underline{k}, \underline{k}', \underline{v}) , \quad (48)$$

respectively. Here

$$G(\underline{k}, \underline{k}', \underline{v}) \equiv \int d\tau \tau^2 h_{\omega}(-\tau, \underline{k}') h_v(-\tau, \underline{k}', \underline{v}) h_{\omega}(-\tau, \underline{k}'') h_0(-\tau, \underline{k}'') h'(-\tau, \underline{k}'') \quad (49)$$

is a modulation function having a streaming by h_v and a memory-loss by $h_0 h'$.

In strong turbulence, the memory-loss predominates over the streaming, and renders the modulation function to become independent of \underline{v} as in

$$G(\underline{k}'') = \int_0^\infty d\tau \tau^2 h_\omega(-\tau, \underline{k}') h_0(-\tau, \underline{k}') h'(-\tau, \underline{k}''). \quad (50)$$

Thus we obtain an eddy viscosity

$$K'_{jn} = \int d\underline{k}'' \langle j'_{jn}(\underline{k}'') \rangle G(\underline{k}''). \quad (51)$$

and transform (48) into

$$T^{(0)} \Big|_{\text{large } k} = K'_{jn} \int d\underline{k}' \frac{k'_j k'_n}{j' n'} \int d\underline{v} v_i \varphi_i^{(0)}(\underline{k}', \underline{v}). \quad (52)$$

The integrations herein involved are:

$$\begin{aligned} \int d\underline{v} v_i \varphi_i^{(0)}(\underline{k}', \underline{v}) &= \int d\underline{v} v_i \chi \langle u_i^{(0)}(\underline{k}') f^{(0)}(-\underline{k}', \underline{v}) \rangle \\ &= \chi \langle u_i^{(0)}(\underline{k}') u_i^{(0)}(-\underline{k}') \rangle \end{aligned} \quad (53a)$$

$$\int d\underline{k}' \chi \langle u_i^{(0)}(\underline{k}') u_i^{(0)}(-\underline{k}') \rangle = \langle u_i^{(0)}(t, \underline{x}) u_i^{(0)}(t, \underline{x}) \rangle, \quad (53b)$$

and

$$\begin{aligned} \int d\underline{k}' \frac{k'_j k'_n}{j' n'} \int d\underline{v} v_i \varphi_i^{(0)}(\underline{k}', \underline{v}) &= \int d\underline{k}' \frac{k'_j k'_n}{j' n'} \chi \langle u_i^{(0)}(\underline{k}') u_i^{(0)}(-\underline{k}') \rangle \\ &= \langle \nabla_j u_i^{(0)} \nabla_n u_i^{(0)} \rangle \equiv R_{jn}^{(0)}. \end{aligned} \quad (54)$$

Hence, by substitution into (52), we obtain the transfer function

$$T^{(0)}_{jr} \Big|_{\text{large } k} = K'_{jr} R^{(0)}_{rj} \quad (55)$$

in the form of the product of the eddy viscosity K'_{jr} by the vorticity function $R^{(0)}_{rj}$. Here, by isolating an eddy viscosity, we have found a cascade transfer of the gradient type.

Case (b). Small k

The small values of k imply an even smaller k' . This extremely small k' , as compared to k'' , suppresses all roles of $\hat{M}(-\tau, k', v)$ before $\hat{M}(-\tau, k'', v)$, rendering

$$\hat{M}(-\tau, k', v) \cong 1. \quad (56)$$

Thus we transform (39) into

$$N^{(0)}_{jr} \Big|_{\text{small } k} = -k''_j k''_r \tau^2 \int d\nu \, \nu_i \varphi_i^{(0)}(k', \nu) \hat{M}(-\tau, k'', \nu). \quad (57)$$

The factor $k''_j k''_r$ emphasizes large values of k'' so that we can write

$$\hat{M}(-\tau, k'', \nu) \approx h_\omega h_\nu \bar{h} h^{(0)} h' \quad (58)$$

as an approximation. When this function is substituted

into (57) and subsequently integrated with respect to τ , following (40), the contribution from h_v as a streaming is again negligible as compared to the memory-loss for large k'' in strong turbulence. It follows a modulation function

$$\begin{aligned} G_{\sim}(k'') &= \int_0^{\infty} d\tau \tau^2 \hat{M}(-\tau, k'') \\ &\approx \int_0^{\infty} d\tau \tau^2 h_{\omega}(-\tau, k'') h_0(-\tau, k'') h'(-\tau, k'') \end{aligned} \quad (59)$$

for the control of the transfer function (40), yielding:

$$T^{(0)} \Big|_{\text{small } k} = \int d\tilde{k}'' k''_{\sim} k''_f \langle s'_{j\sim}(k'') \rangle G_{\sim}(k'') \int d\tilde{k}' \int d\tilde{v} v_i \varphi_i^{(0)}(k', v). \quad (60)$$

Here the two integrals with respect to \tilde{k}' and \tilde{k}'' become separated into an energy

$$\langle u^{(0)2} \rangle_{\sim} = \int d\tilde{k}' \int d\tilde{v} v_i \varphi_i^{(0)}(k', v) \quad (61a)$$

by (56) and (30b), and a frequency

$$\Omega' = \int d\tilde{k}'' k''_{\sim} k''_f \langle s'_{j\sim}(k'') \rangle G_{\sim}(k''). \quad (61b)$$

called the rate of damping in the collisionless damping of the energy. Hence the transfer function becomes:

$$T^{(0)} \Big|_{\text{small } k} = \Omega' \langle u^{(0)2} \rangle_{\sim}. \quad (62)$$

Since the transfer of energy is not of the gradient type, we cannot isolate an eddy viscosity as was the case of large k in (55).

We may consider

$$\langle u_{\sim}^{(0)2} \rangle \{ \} \quad \text{and} \quad K'_{jr} \{ \} \quad (63)$$

as integral operators which emphasize large and small values of k by their respective integrations

$$\int_0^k dk' \quad \text{and} \quad \int_k^\infty dk'' ,$$

and write

$$R_{nj}^{(0)} = \langle u_{\sim}^{(0)2} \rangle \{ k'_n k'_j \} \quad \text{and} \quad \Omega' = K'_{jr} \{ k''_n k''_j \} . \quad (64)$$

In this manner, the two transfer functions (55) and (62) become

$$T^{(0)}_{\text{large } k} = K'_{jr} \langle u_{\sim}^{(0)2} \rangle \{ k'_n k'_j \} \quad (65)$$

and

$$T^{(0)}_{\text{small } k} = K'_{jr} \{ k''_n k''_j \} \langle u_{\sim}^{(0)2} \rangle . \quad (66)$$

For large and small k , the operators (63) pick up

$$k'_n k'_j \quad \text{and} \quad k''_n k''_j ,$$

respectively, from the sum $k'_r k'_j + k''_r k''_j$ as appearing in

$$\partial_j \partial_n \hat{M}(k'_j) \quad \text{and} \quad \partial_j \partial_n \hat{M}(k''_j)$$

of the general transfer function (40). Thus the formula (40) is a general transfer function with the special forms (65) and (66).

VI. DERIVATION OF THE KOLMOGOROFF LAW FROM THE SCALED KINETIC EQUATION

Although the Kolmogoroff law¹³ of the inertia turbulence has been derived by many investigators using diverse methods as seen in a recent review²¹ of the diagram technique²² in conjunction with the direct interaction approximation²³, it is for the first time that the analytical derivation of the Kolmogoroff law and its numerical coefficient is obtained from the scaled kinetic equation of turbulence. Our kinetic method is generally valid for inertia and non-inertia turbulence.

We assume an isotropic and homogeneous turbulence, and that the field

$$\underline{E} = - \frac{1}{\rho} \nabla p \quad (67)$$

comes from the pressure gradient ∇p in an incompressible fluid of constant density ρ . The shear and the buoyancy are absent. The condition of incompressibility, as applied

to the Navier-Stokes equation (1) gives the relation

$$\nabla \cdot \underline{\underline{E}} = \nabla \nabla : \underline{\underline{u}} \underline{\underline{u}} \quad (68)$$

between $\underline{\underline{E}}$ and $\underline{\underline{u}}$, and therefore also between their spectral distributions $S(k)$ and $F(k)$. We have

$$\frac{1}{2} \langle \underline{\underline{E}}^2 \rangle = \int_0^\infty dk S(k), \quad \frac{1}{2} \langle \underline{\underline{u}}^2 \rangle = \int_0^\infty dk F(k), \quad \frac{1}{2} \langle p^2 \rangle = \int_0^\infty dk \Pi(k),$$

and find the relations (69)

$$\begin{aligned} S(k) &= \alpha F(k) R^{(0)}(k) \\ \Pi(k) &= \rho^2 k^{-2} S(k), \end{aligned} \quad (70)$$

where $R^{(0)} \equiv R_{ii}^{(0)}$ is the vorticity function, from (54), and the numerical coefficient is determined to be 19,20

$$\alpha = \frac{4}{9}. \quad (71)$$

The details of the calculation are omitted.

In the inertia subrange, the energy balance in the spectral form is^{13,14}

$$T^{(0)} = \varepsilon, \quad (72)$$

describing a constant cascade of energy transfer across the spectrum at the rate of energy dissipation ε . Now the trans-

fer function, as found in (55), can be written in the isotropic form:

$$T^{(0)} = K' R^{(0)}, \quad (73)$$

where

$$K' = \frac{2}{3} \int_k^\infty dk'' S(k'') G(k'') \quad (74)$$

is the trace of the eddy viscosity tensor (51), and the modulation function is given by (50), with $\omega_k = 0$.

From (9), (18a) and (19), we rewrite the modulation function (50) in the form:

$$G(k'', K') = \int_0^\infty d\tau \tau^2 \exp(-m^{(0)} \tau^4 - \omega_K' \tau). \quad (75)$$

It contains the decay of the memory at a rate

$$m^{(0)} = \left[\frac{1}{24} k''^2 \langle \underline{E}^{(0)2} \rangle \right]^{\frac{1}{4}} \quad (76a)$$

and the cut-off of the memory at a time $\omega_K'^{-1}$ such that

$$\omega_K' = k''^2 K'. \quad (76b)$$

The integral is evaluated by an interpolation, giving a modulation function

$$G(k'', K') = \left\{ \left[\frac{1}{3} \Gamma\left(\frac{7}{4}\right) \right]^{-\frac{1}{3}} m^{(0)} + 2^{-1/3} \omega_K' \right\}^{-3}. \quad (77)$$

Hence we determine the eddy viscosity by the following integral equation:

$$K' = \frac{2}{3} \int_k^\infty dk'' S(k'') G[k'', K'(k'')] . \quad (78)$$

When the spectrum $S(k)$ is converted into the spectrum $F(k)$ by the relation (70), we find the solution of (73), giving the spectra

$$\begin{aligned} F(k) &= A \epsilon^{2/3} k^{-5/3}, \quad \text{with } A = 1.7 \\ S(k) &= \frac{2}{3} A^2 \epsilon^{4/3} k^{-1/3} \\ \Pi(k) &= \frac{2}{3} A^2 \rho^2 \epsilon^{4/3} k^{-7/3}, \end{aligned} \quad (79)$$

of velocity, field and pressure fluctuations, respectively. The details of calculations are omitted.

VII. DISCUSSION

The micro-dynamical state of turbulence can be described by a nonlinear and stochastic hydrodynamical system which may be homogeneous or inhomogeneous. The Navier-Stokes system of equations is inhomogeneous, as in (1), or briefly as

$$(\partial_t + L) \hat{u} = \hat{E}, \quad \nabla \cdot \hat{u} = 0, \quad \text{with } L \equiv \hat{u} \cdot \nabla - \nu \nabla^2. \quad (80)$$

The inhomogeneous system in the physical space may be converted into a homogeneous equation in the phase space, as

$$(\partial_t + L) \hat{N}(t, x, v) = 0, \quad \text{with } L \equiv v \cdot \nabla - \nu \nabla^2 + \hat{E} \cdot \partial, \quad (81)$$

called the "master equation". It is equivalent to the inhomogeneous system (80), provided

$$\hat{N}(t, x, v) = \int \delta[v - \hat{u}(t, x)] \cdot \quad (82)$$

Other homogeneous systems may be:

(i) the Burgers equation

$$(\partial_t + L) u = 0, \quad \text{with } L \equiv u \frac{\partial}{\partial x} - \nu \frac{\partial^2}{\partial x^2}, \quad (83a)$$

(ii) the Korteweg-DeVries equation

$$(\partial_t + L) u = 0, \quad \text{with } L \equiv u \frac{\partial}{\partial x} + \lambda \frac{\partial^3}{\partial x^3} \quad (83b)$$

(iii) the equation of geostrophic turbulence

$$(\partial_t + L) \nabla \times u = 0, \quad \text{with } L \equiv u \cdot \nabla - \nu \nabla^2. \quad (83c)$$

From the conceptual viewpoint, being given the micro-dynamical state of turbulence, the question may arise as to what treatment one should choose: the kinetic treatment or the hydrodynamic treatment. Since a homogeneous partial differential equation can describe the exact orbit dynamics by the differential perturbation operator L alone, there is an essential advantage to keep a homogeneous system, either in the hydrodynamic representation (83) or in the kinetic representation (81). For this reason, we have chosen the kinetic method which, moreover, suppresses all nonlinearities caused by the velocity since the velocity is an independent variable here. The only nonlinearity that subsists is from $\hat{E} \cdot \partial \hat{N}$ in (81). This entails a Reynolds stress in the phase space, the evolution of which requires the knowledge of the Lagrangian correlation of \hat{E} -field fluctuations and a theory of transport. The approach to equilibrium and the loss of memory constitute the basis for the closure of turbulence.

There are two Lagrangian correlations involved: one is the correlation of \hat{E} -field fluctuations as stated above, and the other is the correlation between the velocity fluctuations and the fluctuating distribution of velocities. The latter arises because our kinetic equation (5) of turbulence has a long memory. Since each Lagrangian correlation entails a memory, our kinetic theory of transport finds two memory-loss functions. Their competitive roles

and their cut-off are the main issues of our analysis. The difficulties are minimized by introducing a scaling procedure which dispenses from our involvement with the two-point distribution function. Then the scaled probabilities of transition determine the scaled orbit functions. The scaling permits the distinction between the macro- and micro-variances of path fluctuations which are associated with the unmaturred (i.e. small time) and the maturred (i.e. large time) properties of transport. Otherwise, the unscaled variance would appear as an external parameter.

Our results indicate that the transfer function takes a gradient form for small scale turbulence, isolating an eddy viscosity, and a form of collisionless damping without gradient for large scale turbulence, isolating a rate of collisionless damping.

We can write the equation of energy (26) in the form:

$$\frac{\partial}{\partial t} \int_0^k dk' F(k') = P^{(o)} - B^{(o)} - T^{(o)} - \epsilon^{(o)}, \quad (84)$$

with the transport functions: production $P^{(o)}$, loss $B^{(o)}$, transfer $T^{(o)}$ and dissipation $\epsilon^{(o)}$. In statistical equilibrium the left-hand side is independent of k , so that, after a differentiation with respect to k , we have the spectral balance

$$\dot{T}^{(0)} = -(\dot{P}' - \dot{B}') \quad (85)$$

for scales larger than that of the inertia turbulence.

Note that ϵ ceases to be the governing parameter here.

This means that

$$\dot{T}^{(0)} > 0 \quad \text{in a net production} \quad (86a)$$

$$\dot{T}^{(0)} < 0 \quad \text{in a net loss.} \quad (86b)$$

The change of signs in (86a) and (86b) refers to a direct cascade (i.e. a cascade transfer toward large wavenumbers) and to an inverse cascade (i.e. a cascade transfer toward small wavenumbers), respectively. The transfer function in the form of collisionless damping is preferred for the description of the inverse cascade. The arbitrary use of the transfer function in the form of gradient transport would result in a negative spectrum.

ACKNOWLEDGEMENTS. This work was sponsored by a grant from the National Aeronautics and Space Administration.

REFERENCES

1. A.S. Monin, Dokl. Akad. Nauk SSSR 177, 1036 (1967) [Sov. Phys. Doklady 12, 1113 (1968)].
2. A.S. Monin, Priklad. Mat. Mekh. 31, 1057 (1967) [Appl. Math. and Mech. 31, 1057 (1967)].
3. T.S. Lundgren, Phys. Fluids 10, 969 (1967).
4. T.S. Lundgren, Phys. Fluids 12, 485 (1969).
5. F.R. Ulinich and B. Ya. Lyubimov, Zh. Eksp. Teor. Fiz. 55, 951 (1968) [Sov. Phys. JETP 28, 494 (1969)].
6. V.M. Ievlev, Mez. Zhid. Gaza, No. 1 (1970).
7. V.M. Ievlev, Dokl. Akad. Nauk SSSR 208, 1044 (1973) [Sov. Phys. Doklady 18, 117 (1973)].
8. C.M. Tchen, Comptes Rendus Acad. Sci. Paris 286A, 605 (1978).
9. C.M. Tchen, "Kinetic Basis of Cascade Transfer in Turbulence", (manuscript).
10. T.S. Lundgren and Y.B. Pointin, Phys. Fluids 19, 355 (1976).
11. J. Weinstock, Phys. Fluids 21, 887 (1978).
12. J. Weinstock, Phys. Fluids 19, 1702 (1976).
13. A.N. Kolmogoroff, Compt. Rendus Acad. Sci. U.R.S.S. 30, 301 (1941).
14. W. Heisenberg, Z. Physik 124, 628 (1948).
15. J. Weinstock, J. Atmos. Sci. 37, 1542 (1980).
16. J. Weinstock, J. Atmos. Sci. 35, 634 (1978).
17. J. Weinstock, J. Geophys. Res. 81, No.4, 633 (1976).
18. C.M. Tchen, Versl. Ned. Akad. Wetensch., Afd. Natuurkunde, Amsterdam, 53, 400 (1944).
19. C.M. Tchen, "A Probability Theory of Retrograde Transition for the Space-Time Transformation in Turbulent Diffusion". Manuscript accepted for publication in Proc. Int. Conf. of Fluid Dynamics and Applications, Greece, 1981.

20. C.M. Tchen, "A Kinetic Method of Turbulence". Manuscript accepted for publication in Proc. Int. Conf. of Fluid Dynamics and Applications, Greece, 1981.
21. E.A. Kuznetsov and V.S. L'vov, Physica 2D, 203 (1981).
22. H.W. Wyld, Ann. Phys. 14, 143 (1961).
23. R.F. Kraichnan, Phys. Fluids 7, 575 (1965); 8, 2219 (1965).

CHAPTER C: QUARTER THREE

Group-Scaling Theory for the Enstrophy Transfer in Two-Dimensional Turbulence

C.M. Tchen

The Graduate Center and The City College of
The City University of New York, N.Y. 10031

ABSTRACT

Since the detailed interactions between the individual modes contain too many minute details unnecessary for a statistical theory, we scale the modes into groups which can more easily decipher the governing transport processes and the statistical characteristics. An equation of vorticity transport describes the microdynamical state of two-dimensional, isotropic and homogeneous, geostrophic turbulence. By the group-scaling, we derive the equation of evolution of the macro-vorticity in the form of the Fokker-Planck equation with memory. The eddy diffusivity and the enstrophy transfer relax to equilibrium through the memory-loss by turbulent dispersions. The memory-loss is analyzed by developing a theory of probability of retrograde transition in the scaled form. We find that the eddy diffusivity contains two time scales, characteristic of

the small-time scattering and the large-time diffusion. In the special cases, our formula can be degenerated to an empirical form suggested by Heisenberg when the scattering prevails, and to a form derived earlier by Tchen when the diffusion is dominant. In the inertia subrange, as governed by the enstrophy cascade, the spectral law $F_u = C \varepsilon_\zeta^{2/3} k^{-3}$ is derived with a numerical coefficient $C = 2.59$.

I. INTRODUCTION

The micro-dynamical state of turbulence can be described by the equations in the following forms:

- (a) a non-conservative (or inhomogeneous) equation in the physical space,
- (b) a conservative (or homogeneous) equation in the physical space,,
- (c) a conservative equation in the phase space.

The form (a) contains a stirring force, which may be the pressure gradient, the buoyancy, or the random noise. The Navier-Stokes equation falls under the form (a). The two-dimensional geostrophic turbulence also belong to the form (a), if the turbulence is driven by a random vorticity source, and will fall into the form (b) if such a source is absent. The one-dimensional Burgers equation and the Korteweg-DeVries equation also falls into the form (b).

It is to be noted that the non-conservative equation (a) in the physical space can be converted into a conservative equation (c) in the phase space by regarding the stirring field as an advection. Therefore the problems of the conservative system should be the main objective of a statistical theory. With this emphasis in mind, we develop a statistical theory, using the scaling procedure to study the mode-couplings and the transport property, and relying upon the memory-loss to obtain the

closure of turbulence. A kinetic theory¹ on the basis (c) has been developed by Tchen for the Navier-Stokes turbulence (a). Presently, we apply the scaling method to the two-dimensional, isotropic and homogeneous, geostrophic turbulence.

In view of the many applications to the meso-scale flows in the atmosphere and the availability of numerical computations²⁻⁷, the problem of geostrophic turbulence has attracted many investigators⁸⁻¹⁰, see a review by Rhines¹¹. The geostrophic turbulence is characterized by an enstrophy transfer from large scales toward small scales, yielding a spectrum of velocity fluctuations following the k^{-3} law. Most theories analyzed the triad interactions¹²⁻¹⁸.

The present theory simplifies the interactions by scaling a fluctuation into three groups, representing the evolution, the eddy transport and the relaxation. Our purpose is to derive analytically the enstrophy transfer function, the eddy transport coefficient and the spectral distribution. Upon a generalization to a kinetic method (c), as is equivalent to the non-conservative micro-dynamics (a), we expect to find:

- (i) the spectral law $k^{-5/3}$ in a direct cascade of energy transfer at large k ,
- (ii) the spectral law $k^{-5/3}$ in an inverse cascade of energy transfer at small k ,

(iii) the spectral law k^{-4} from an enstrophy transfer in a geostrophic turbulence that is driven by randomly distributed vorticity sources. These results will be reported on a separate occasion.

II. SCALED VORTICITY EQUATION OF TURBULENCE

The two-dimensional incompressible flow in the horizontal plane is governed by the following equations of motion:

$$(\partial_t + L) \hat{\xi}(t, x) = 0 \quad (1a)$$

and

$$\nabla \cdot \hat{u} = 0 \quad (1b)$$

Here \hat{u} is the velocity,

$$\hat{\xi} \equiv \nabla \times \hat{u} = (0, 0, \hat{\xi}) \quad (2)$$

is the scalar vorticity,

$$L \equiv \hat{u} \cdot \nabla - \nu \nabla^2 \quad (3)$$

is the differential operator, and ν is the kinematic viscosity. By the condition (1b), the pressure gradient, the vertical buoyancy force, and the vertical Coriolis parameter are eliminated. The system (1) describes the

micro-dynamical state of geostrophic turbulence.

The scaling may be made by means of the operator

$$\bar{A} = \langle \rangle \quad (4a)$$

of global average and the fluctuation operator

$$\tilde{A} = 1 - \bar{A} \quad (4b)$$

giving the deviation from the global average. The fluctuation operator

$$\tilde{A} = A^{(0)} + A' \quad (4c)$$

is subsequently divided into the operator $A^{(0)}$ of macro-fluctuation and the operator A' of micro-fluctuation.

For the sake of convenience, we introduce an operator

$$A_0 \equiv \bar{A} + A^{(0)} = 1 - A', \quad (4d)$$

giving an accumulated macro-group. Thus we have:

$$\tilde{A} \hat{u} = u, \quad u = u^{(0)} + u' \quad (5a)$$

and

$$\tilde{A} \hat{\zeta} = \zeta, \quad \zeta = \zeta^{(0)} + \zeta' \quad (5b)$$

The macro-fluctuation and the micro-fluctuation can instantaneously overlap in their wavenumber components, but their statistical functions are distinctly separated by a wavenumber variable k , and by their durations of correlation

$$\tau_c^{(0)} > \tau_c' \quad (6)$$

in the increasing degrees of incoherence.

By applying $A^{(0)}$ and A' to (1a), we obtain the following scaled system:

$$(\partial_t + A^{(0)}L) \zeta^{(0)} = - \nabla \cdot A^{(0)} \underline{u}' \zeta' \quad (7a)$$

$$(\partial_t + L) \zeta' = - \underline{u}' \cdot \nabla \zeta^{(0)} + \nabla \cdot A' \underline{u}' \zeta'. \quad (7b)$$

By a formal integration of (7b) and a substitution into (7a), we eliminate ζ' to obtain the equation of evolution of the macro-vorticity:

$$(\partial_t + A^{(0)}L) \zeta^{(0)}(t, \underline{x}) = \nabla \cdot \underline{K}' \{ \nabla \zeta^{(0)} \} \quad (8)$$

where

$$\underline{K}' = \int_0^{t \rightarrow \infty} d\tau \langle \underline{u}'(t, \underline{x}) U(t, t-\tau) \underline{u}'(t-\tau) \rangle \quad (9)$$

is the eddy diffusivity, and $U(t, t-\tau)$ is the propagator,

or the operator of evolution along an orbit that is perturbed according to (3). Without altering the value of the integral, the upper limit may be put to infinity, by the scale separation (6), giving a matured (i.e. large-time) diffusivity. It is to be noted that the diffusivity enters as an integral operator $\tilde{K}'\{\}$ in (8), so that this scaled vorticity equation takes the form of an integro-differential equation, and not the customary Fokker-Planck differential equation.

When we multiply the vorticity equation (8) by $\zeta^{(0)}$ and average, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle \zeta^{(0)2} \rangle + \frac{1}{2} \nabla \cdot \langle (\tilde{u} + \tilde{u}^{(0)}) \zeta^{(0)2} \rangle \\ = \langle \zeta^{(0)} \nabla \cdot \tilde{K}'\{ \cdot \nabla \zeta^{(0)} \} \rangle + \nu \langle \zeta^{(0)} \nabla^2 \zeta^{(0)} \rangle. \end{aligned} \quad (10)$$

Here we have retained the kinematic viscosity in the differential operator L .

In a homogeneous turbulence, the second term on the left-hand side vanishes, and the last term on the right-hand side becomes

$$\nu \langle \zeta^{(0)} \nabla^2 \zeta^{(0)} \rangle = - \nu \langle \tilde{\psi}^{(0)} \rangle \equiv - \xi^{(0)} \text{ with } \tilde{\psi}^{(0)} \equiv \nabla \zeta^{(0)}. \quad (11a)$$

In an analogous way, we can write

$$K' \langle \zeta^{(0)} \nabla^2 \zeta^{(0)} \rangle = -K' \langle \psi^{(0)2} \rangle, \quad (11b)$$

and find the first term on the right-hand side of (10) to be

$$- \langle \zeta^{(0)} \nabla \cdot K' \{ \nabla \zeta^{(0)} \} \rangle = K' \{ \langle \psi^{(0)}(t) \cdot \psi^{(0)}(t-\tau) \rangle \} = T_{\zeta}^{(0)}. \quad (11c)$$

Here the diffusivity remains to be an integral operator with its time integration extended to the Lagrangian correlation

$$\langle \psi^{(0)}(t) \cdot \psi^{(0)}(t-\tau) \rangle. \quad (11d)$$

The diffusivity is written in the form of a trace. The function $\varepsilon_{\zeta}^{(0)}$ is the rate of molecular dissipation of the enstrophy $\frac{1}{2} \langle \zeta^{(0)2} \rangle$. The function $T_{\zeta}^{(0)}$ represents the mode-coupling and is called the "enstrophy transfer".

In terms of the dissipation function and the transfer function, the equation of spectral balance (10) is reduced to the form:

$$\frac{1}{2} \frac{d}{dt} \langle \zeta^{(0)2} \rangle = - T_{\zeta}^{(0)} - \varepsilon_{\zeta}^{(0)}. \quad (12)$$

It represents the time decay of the mean vorticity or enstrophy

$$\frac{1}{2} \langle \zeta^{(0)2} \rangle = \int_0^k dk' F_{\zeta}(k'), \quad (13)$$

which has a spectral distribution $F_{\zeta}(k)$ of ζ -fluctuations.

III. TRANSPORT PHENOMENA IN TURBULENCE

A. Lagrangian-Eulerian transformation in the scaled form

The Lagrangian velocity follows a trajectory that is governed by the dynamical equation:

$$\frac{d\hat{x}(t_1)}{dt_1} = \hat{u}(t_1) \quad (14a)$$

or the orbit equation

$$\frac{d\hat{\ell}(-\tau)}{d\tau} = -\hat{u}(-\tau), \quad (14b)$$

where the perturbed path

$$\hat{\ell}(-\tau) = \bar{\ell}(-\tau) + \underline{\ell}(-\tau) \quad (14c)$$

consists of a mean path $\bar{\ell}(-\tau)$ and a path fluctuation $\underline{\ell}(-\tau)$. In the following calculations of the path perturbations, we shall neglect the effect of the kinematic viscosity ν as compared to the larger effect from the eddy motions.

By means of the Fourier transformation, we can write the Lagrangian and Eulerian correlations in the following

forms:

$$\begin{aligned} W_L'(\tau) &= \langle \tilde{u}'(\tilde{t}, \tilde{x}) \tilde{U}(\tilde{t}, \tilde{t}-\tau) \tilde{u}'(\tilde{t}-\tau) \rangle \\ &= \int d\tilde{k}'' \tilde{h}_\omega(-\tau, \tilde{k}'') \langle \tilde{s}_u'(\tilde{k}'') e^{i\tilde{k}'' \cdot \hat{\tilde{l}}(-\tau)} \rangle \end{aligned} \quad (15)$$

and

$$\begin{aligned} W_E'(\tau) &= \langle \tilde{u}'(\tilde{t}, \tilde{x}) \tilde{u}'(\tilde{t}-\tau, \tilde{x}-\tilde{\bar{l}}) \rangle \\ &= \int d\tilde{k}'' \tilde{h}_\omega(-\tau, \tilde{k}'') \langle \tilde{s}_u'(\tilde{k}'') \rangle \tilde{h}(-\tau, \tilde{k}''), \quad \tilde{h}(-\tau, \tilde{k}'') = e^{i\tilde{k}'' \cdot \tilde{\bar{l}}(-\tau)}, \end{aligned} \quad (16)$$

respectively. Here $\tilde{s}_u'(\tilde{k}'')$ is a fluctuating intensity in the \tilde{k} -space, such that

$$\langle \tilde{s}_u'(\tilde{k}'') \rangle = \chi \langle \tilde{u}'(\tilde{k}'') \tilde{u}'(-\tilde{k}'') \rangle \quad (17a)$$

is the spectral function, and

$$d \operatorname{tr} \int d\tilde{k}'' \langle \tilde{s}_u'(\tilde{k}'') \rangle = \langle \tilde{u}'(\tilde{t}, \tilde{x}) \cdot \tilde{u}'(\tilde{t}, \tilde{x}) \rangle = \int_{\tilde{k}}^{\infty} d\tilde{k}'' F_u(\tilde{k}'') \quad (17b)$$

is the mean intensity in the \tilde{x} -space. The coefficient

$$\chi = (\pi/\chi)^d$$

is the factor of truncation, when the Fourier transformation is truncated within a length interval $2X$ in two dimensions, i.e. $d=2$. The Fourier transformation has a

Fourier kernel

$$h_{\omega}(-\tau, \tilde{k}'') \equiv e^{i \omega_k(\tilde{k}'') \tau}, \quad (18)$$

with a frequency $\omega_k = \omega_k(\tilde{k}'')$ that arises from a dispersion relation, e.g. as relating to an internal gravity wave!¹⁹

The integrations in (15)-(17) and in the following are understood to extend from $-\infty$ to ∞ .

The correlation in the integrand of (15) can be written as

$$\begin{aligned} \left\langle \tilde{J}'_u(\tilde{k}'') e^{i \tilde{k}'' \cdot \hat{\ell}(-\tau)} \right\rangle &= \left\langle \tilde{J}'_u(\tilde{k}'') (A_o + A') e^{i \tilde{k}'' \cdot \hat{\ell}(-\tau)} \right\rangle \\ &= \left\langle \tilde{J}'_u(\tilde{k}'') A_o e^{i \tilde{k}'' \cdot \hat{\ell}(-\tau)} \right\rangle + \left\langle \tilde{J}'_u(\tilde{k}'') A' e^{i \tilde{k}'' \cdot \hat{\ell}(-\tau)} \right\rangle, \end{aligned} \quad (19)$$

consisting of a mutually independent part that can be factorized as

$$\begin{aligned} \left\langle \tilde{J}'_u(\tilde{k}'') A_o e^{i \tilde{k}'' \cdot \hat{\ell}(-\tau)} \right\rangle &= \left\langle \tilde{J}'_u(\tilde{k}'') \right\rangle \left\langle A_o e^{i \tilde{k}'' \cdot \hat{\ell}(-\tau)} \right\rangle \\ &\equiv \left\langle \tilde{J}'_u(\tilde{k}'') \right\rangle \bar{A} A_o e^{i \tilde{k}'' \cdot \hat{\ell}(-\tau)}, \end{aligned} \quad (20a)$$

and a correction

$$\left\langle \tilde{J}'_u(\tilde{k}'') A' e^{i \tilde{k}'' \cdot \hat{\ell}(-\tau)} \right\rangle \quad (20b)$$

accounting for the mutual dependence. We find that this correction is not too important for the small-time scattering and the large-time diffusion that are of interest in our problem. The mutually independent part contains an average

$$\bar{A} A_0 e^{i \underline{k}'' \cdot \hat{\underline{l}}(-\tau)} \quad (21)$$

that is calculated by the probabilities of transition in the scalings by A_0 and \bar{A} . The derivation of the probability functions and the calculations of the average are presented in Appendix A, finding

$$\bar{A} A_0 e^{i \underline{k}'' \cdot \hat{\underline{l}}(-\tau)} = \bar{h}(-\tau, \underline{k}'') h^{(0)}(-\tau, \underline{k}'') h'(-\tau, \underline{k}''). \quad (22)$$

On the right-hand side appear the components of the orbit functions, or "orbit components". The mean path gives the component

$$\bar{h}(-\tau, \underline{k}'') = e^{i \underline{k}'' \cdot \bar{\underline{l}}(-\tau)} \quad (23a)$$

The fluctuations give the components

$$h^{(0)}(-\tau, \underline{k}'') = \langle e^{i \underline{k}'' \cdot \underline{l}^{(0)}(-\tau)} \rangle = \exp(-m^{(0)2} \tau^2) \quad (24a)$$

$$h'(-\tau, \underline{k}'') = \langle e^{i \underline{k}'' \cdot \underline{l}'(-\tau)} \rangle = \exp(-\omega_K'^2 \tau^2), \quad (24b)$$

in the unmatured and matured dispersions, respectively, of time scales $m^{(o)-1}$ and $\omega_D'^{-1}$, such that

$$\begin{aligned} m^{(o)2} &= \frac{1}{2} k'' k'' : \langle u_{\sim}^{(o)} u_{\sim}^{(o)} \rangle \big|_{k''} \\ &\cong d^{-2} k''^3 F_u(k'') \end{aligned} \quad (25a)$$

$$\omega_K' = k'' k'' : K_{\sim}' \cong k''^2 K'. \quad (25b)$$

The details of the calculation are given in (A20)-(A24) of Appendix. The function $F_u(k'')$ is the spectral distribution of velocity fluctuations, as described by (A19) in Appendix.

If we lump the orbit components caused by the fluctuations into the function

$$M \equiv h^{(o)} h', \quad (26)$$

called the "memory-loss function", we can write the Lagrangian-Eulerian relationship in the form:

$$W_L'(\tau) \underset{\sim}{=} W_E' \left\{ M(-\tau, k'') \right\}. \quad (27)$$

This is a nonlinear integral equation for $W_L'(\tau)$, since M contains W_L' through h' , ω_K' and (9). Note that the Eulerian correlation is governed by the mean path $\bar{x}_{\sim}(-\tau)$, while the Lagrangian correlation is governed by the mean path

and the path fluctuation $\tilde{x}(-\tau)$.

B. Relaxation time

Upon substituting (27) into (9), we find the diffusivity tensor and its trace in the form:

$$\begin{aligned} K'_{\tilde{z}} &= \int d\tilde{k}'' \left\langle \tilde{s}'_{\tilde{u}}(\tilde{k}'') \right\rangle \int_0^{\infty} d\tau M(-\tau, \tilde{k}'') \\ &= \int d\tilde{k}'' \left\langle \tilde{s}'_{\tilde{u}}(\tilde{k}'') \right\rangle G_K(\tilde{k}'') . \end{aligned} \quad (28a)$$

$$\begin{aligned} K' &= \text{tr}_{\tilde{z}} K'_{\tilde{z}} = \int d\tilde{k}'' \text{tr}_{\tilde{z}} \left\langle \tilde{s}'_{\tilde{u}}(\tilde{k}'') \right\rangle G_K(\tilde{k}'') \\ &= \frac{2}{d} \int_k^{\infty} d\tilde{k}'' F_{\tilde{u}}(\tilde{k}'') G_K(\tilde{k}'') . \end{aligned} \quad (28b)$$

It is seen that the diffusivity approaches its equilibrium in a relaxation time

$$\begin{aligned} G_K(\tilde{k}'') &= \int_0^{\infty} d\tau M(-\tau, \tilde{k}'') \\ &= \int_0^{\infty} d\tau \exp\left(-m^{(0)2} \tau^2 - \omega_K' \tau\right) , \end{aligned} \quad (29a)$$

or, approximately,

$$G_K(\tilde{k}'') \cong \left[(2/\sqrt{\pi}) m^{(0)2} + \omega_K' \right]^{-1} . \quad (29b)$$

The approximation is based upon the interpolation.

C. Diffusivity

The equation (28b) which determines the diffusivity K' is in reality an integral equation since the relaxation time $G_K(k'')$ is also a function of K' through ω_K' , by (25b).

As an approximation, we calculate the asymptotic values of K' by considering the following two special cases:

case (i) If G_K is controlled by $m^{(0)}$, we find

$$K' = c_K \int_k^\infty dk'' \left[F_u(k'')/k''^3 \right]^{\frac{1}{2}} = K'_m \quad (30)$$

with $c_K = \sqrt{\pi}$ in two and three dimensions.

case (ii) If G_K is controlled by ω_K' , we find

$$K' = \left[\frac{4}{d} \int_k^\infty dk'' k''^{-2} F_u(k'') \right]^{\frac{1}{2}} = K'_\omega \quad (31)$$

As an interpolation, we write:

$$K' = \frac{2}{d} \int_k^\infty dk'' F_u(k'') \left\{ \left[\frac{4}{\pi d^2} k''^3 F_u(k'') \right]^{\frac{1}{2}} + k''^2 K'_\omega \right\}^{-1} \quad (32)$$

It is degenerated into

$$K' = \begin{cases} K'_m, & \text{if } \omega_K' \ll m^{(0)} \\ K'_\omega, & \text{if } \omega_K' \gg m^{(0)}. \end{cases} \quad (33)$$

IV. DOUBLE MEMORY-LOSS

The first memory-loss function $M(-\tau, k'')$ appears in the Lagrangian correlation (27) which produces the diffusivity (32). The second memory-loss function arises from the second Lagrangian correlation (11d), with the same diffusivity in its role of an integral operator as appearing in the transfer function (11c).

In order to derive the structure of the transfer function (11c) in isotropic turbulence, we first write the second correlation function as

$$\langle \tilde{\psi}(t) \cdot \tilde{\psi}(t-\tau) \rangle = \int d\tilde{k}' \langle \tilde{\psi}^{(0)}(\tilde{k}') \rangle M(-\tau, \tilde{k}') \quad (34)$$

in analogy with (27). Here

$$\langle \tilde{\psi}^{(0)}(\tilde{k}') \rangle = \chi \langle \tilde{\psi}^{(0)}(\tilde{k}') \cdot \tilde{\psi}^{(0)}(-\tilde{k}') \rangle \quad (35a)$$

is the spectral intensity of $\tilde{\psi}$ -fluctuations, such that the intensity

$$\int d\tilde{k}' \langle \tilde{\psi}^{(0)}(\tilde{k}') \rangle = \langle \tilde{\psi}^{(0)}(t, \tilde{x}) \cdot \tilde{\psi}^{(0)}(t, \tilde{x}) \rangle$$

can be expressed in terms of $F_{\zeta}(k')$, by

$$\begin{aligned} \int d\tilde{k}' \langle \tilde{\psi}^{(0)}(\tilde{k}') \rangle &= 2 \int_0^k d\tilde{k}' \tilde{k}'^2 \bar{F}_{\zeta}(\tilde{k}') \\ &= \mathcal{R}_{\zeta}^{(0)}. \end{aligned} \quad (36a)$$

Use is made of (11a). Note that the relationship

$$F_{\zeta}(k) = \frac{1}{2} k^2 F_u(k) \quad (36b)$$

between the two spectra follows directly from the definition (2).

Now we apply the integral operator K'_{\sim} to (34) and write the transfer function in the following form

$$T_{\zeta}^{(0)} = K'_{\sim} \left\{ \int d\tilde{k}' \langle J_{\psi}^{(0)}(\tilde{k}') \rangle M(-\tau, \tilde{k}') \right\}, \quad (37a)$$

according to the definition (11c), or in the alternate form

$$T_{\zeta}^{(0)} = \iint d\tilde{k}' d\tilde{k}'' \text{tr} \left\langle J_u^{(0)}(\tilde{k}'') \right\rangle \left\langle J_{\psi}^{(0)}(\tilde{k}') \right\rangle G_T(\tilde{k}', \tilde{k}''), \quad (37b)$$

when K'_{\sim} is written out fully by means of (28b). It is seen that the transfer function approaches its equilibrium at a new relaxation time

$$G_T(\tilde{k}', \tilde{k}'') = \int_0^{\infty} d\tau M(-\tau, \tilde{k}') M(-\tau, \tilde{k}''), \quad (38)$$

with two memories:

$$M(-\tau, \tilde{k}') \quad \text{and} \quad M(-\tau, \tilde{k}''). \quad (39)$$

V. TRANSFER FUNCTION

By the use of (13) and (17), the transfer function (37b) can be rewritten as:

$$T_{\zeta}^{(0)} = \frac{2}{d} \int_k^\infty dk'' F_u(k'') \cdot 2 \int_0^k dk' k'^2 F_{\zeta}(k') G_T(k', k''), \quad (40)$$

where, by definition (38), the relaxation time $G_T(k', k'')$ sums up the contributions from the two competing memory-loss functions (39) under the conditions

$$k' < k < k'', \quad (41)$$

with the properties

$$m^{(0)}(k') \ll m^{(0)}(k'') \text{ and } \omega_K'(k') \ll \omega_K'(k''), \quad (42)$$

so that the role of memory-loss is taken over by $M(-\tau, k'')$, i.e.

$$M(-\tau, k') \cong 1. \quad (43)$$

Thus we can reduce (38) to

$$G_T(k', k'') \cong G_K(k''). \quad (44)$$

and separate the two integrations with respect to the wave-numbers in (40), obtaining the degenerate form

$$T_{\zeta}^{(0)} = \frac{2}{d} \int_k^{\infty} dk'' F_u(k'') G_K(k'') \quad 2 \int_0^k dk' k'^2 F_{\zeta}(k') . \quad (45)$$

or, more briefly,

$$T_{\zeta}^{(0)} = K' R_{\zeta}^{(0)} . \quad (46)$$

in the notations (28b) and (36a).

VI. INERTIA SPECTRUM

In terms of the spectra $F_u(k)$ and $F_{\zeta}(k)$, we can write the spectral balance (12) in the form:

$$\partial_t \int_0^k dk' F_u(k') = - (K' + \nu) 2 \int_0^k dk' k'^2 F_{\zeta}(k') . \quad (47)$$

Since the differential vanishes, i.e.

$$\partial_t F_u(k') = 0$$

in statistical equilibrium, the integral form should read

$$(K' + \nu) 2 \int_0^k dk' k'^2 F_{\zeta}(k') = \xi_{\zeta} . \quad (48)$$

where

$$\xi_{\zeta} = \nu \langle (\nabla^2 \zeta)^2 \rangle \quad (49)$$

is a constant rate of dissipation of the enstrophy. A

fluctuating ε_ζ would enhance the intermittency which we shall not consider at present.

The inertia subrange is described by the spectral balance

$$K' 2 \int_0^k dk' k'^2 F(k') = \varepsilon_\zeta \quad (50)$$

by omitting the effect of molecular dissipation in the left member of (48).

When we use the general formula (32) for K' , and observe the relationship (36b) between the two fluctuations u and ζ , we determine the spectra from (50). The results are:

$$F_u(k) = C \varepsilon_\zeta^{2/3} k^{-3}, \quad C = 2.59 \quad (51)$$

$$F_\zeta(k) = \frac{1}{2} C \varepsilon_\zeta^{2/3} k^{-1}. \quad (52)$$

VII. DISCUSSIONS AND CONCLUSIONS

The problem of turbulence starts with the nonlinear equation describing the micro-dynamical state of turbulence. This equation contains too many minute details which are too complicated and also unnecessary for the statistical theory of turbulence. Past theories either followed the kinetic method and closed at a high order distribution function, or the hydrodynamical method and

closed at a high order correlation. They analyzed the detailed interactions between individual modes. The present theory begins with the coarse-graining procedure by distinguishing between the fluctuations of the macro-group and the fluctuations of the micro-group. It transforms the equation describing the micro-dynamical state into a scaled equation of turbulence, in which the macro-fluctuations evolve in a medium of more random eddies belonging to the micro-group. In this way it analyzes the statistical couplings between groups of modes instead of the detailed couplings between individual modes. For the approach of the transport property to equilibrium, as a closure of turbulence, the memory-loss by turbulent dispersions is essential. We find a scattering by the small-time dispersion and a diffusion by the large-time dispersion in the process of memory-loss and for the derivation of the eddy diffusivity. Our formula (32) of eddy diffusivity can be degenerated to the Heisenberg form (30) when the scattering prevails²⁰, and to the form (31) when the diffusion is dominant. The latter form has been used as an empirical formula in modeling the atmospheric turbulence by Gisina²¹; it has also been derived analytically by Tchen²² from a cascade theory. The transfer of enstrophy is found to cascade down the spectrum toward the large wavenumbers. The spectral law $F_{\zeta}(k) = C\epsilon_{\zeta}^{2/3}k^{-3}$ for the velocity fluctuations and the numerical coefficient

$C = 2.59$, as derived in (51), agrees with the results from the triad interactions by Kraichnan.¹⁴ The nonstationarity of ε_ζ will cause a logarithmic factor due to the intermittency of turbulence, and will be considered on a separate occasion.

APPENDIX. SCALING THEORY OF TRANSITION

We distinguish between a direct transition and a retrograde transition of paths $\hat{\underline{\ell}}(\tau)$ and $\hat{\underline{\ell}}(-\tau)$ in the positive time τ and the negative time $-\tau$, respectively. The probabilities of transition have been investigated by Tchen²³, and were found to be governed by the Fokker-Planck equations. The derivations were based on a phenomenological ground. Here we shall re-examine the retrograde transition by a scaling procedure and on a basis that is consistent with the specified micro-dynamical state of turbulence, i.e. the orbit equation:

$$\frac{d\hat{\underline{\ell}}(-\tau)}{d\tau} = -\hat{\underline{u}}(-\tau) \quad , \quad (\text{A1})$$

see (14b).

The micro-dynamical state of the orbit can be described by the master equation

$$(\partial_\tau + L_\ell) \hat{P}(-\tau, \underline{\ell}) = 0, \quad \partial_\tau \equiv \partial/\partial\tau, \quad (\text{A2})$$

upon the introduction of the δ -function

$$\hat{P}(-\tau, \underline{\ell}) = \delta[\underline{\ell} - \hat{\underline{\ell}}(-\tau)] \quad (\text{A3})$$

and the differential operator

$$L_\ell \equiv -\hat{\underline{u}}(-\tau) \cdot \frac{\partial}{\partial \underline{\ell}} \quad . \quad (\text{A4})$$

We may consider the independent variable \underline{l} to be a random variable, so that \hat{P} becomes a distribution function which satisfies the normalization condition

$$\int d\underline{l} \hat{P} = 1 \quad (\text{A5a})$$

and gives the moment

$$\int d\underline{l} \underline{l} \hat{P} = \underline{\hat{l}}(-\tau). \quad (\text{A5b})$$

It is easy to verify that the moment of (A2) reproduces the orbit equation (A1) identically.

Now we apply the scaling operators A_0 and A' to (A2) and find the following coupled system of equations:

$$(\partial_\tau + A_0 L_{\underline{l}}) P_0 = A_0 \underline{u}'(-\tau) \cdot \frac{\partial P'}{\partial \underline{l}} \quad (\text{A6a})$$

$$(\partial_\tau + A' L_{\underline{l}}) P' = \underline{u}'(-\tau) \cdot \frac{\partial P_0}{\partial \underline{l}}. \quad (\text{A6b})$$

We integrate (A6b) to get

$$P' = \int_0^\infty d\tau' \underline{U}_{\underline{l}}(-\tau, \tau - \tau') \cdot \frac{\partial P_0}{\partial \underline{l}} \quad (\text{A7})$$

and

$$A_0 \underline{u}'(-\tau) \cdot \frac{\partial P'}{\partial \underline{l}} = \frac{\partial}{\partial \underline{l}} \cdot K'_{\underline{l}} \cdot \frac{\partial P_0}{\partial \underline{l}}, \quad (\text{A8})$$

where

$$K'_{\lambda} = \int_0^{\infty} d\tau' \left\langle \tilde{u}'(-\tau) U_{\lambda}(-\tau, -\tau-\tau') \tilde{u}'(-\tau-\tau') \right\rangle \quad (A9)$$

is a diffusivity, and U_{λ} is a propagator or evolution operator in the retrograde transition. The propagator U_{λ} may be considered as the inversion of the differential operator L_{λ} .

When we substitute (A8) into (A6a), we derive the equation of the retrograde transition of the macro-group

$$\partial_{\tau} P_0 = \tilde{u}_0(-\tau) \cdot \frac{\partial P_0}{\partial \lambda} + \frac{\partial}{\partial \lambda} \cdot K'_{\lambda} \cdot \frac{\partial P_0}{\partial \lambda} \quad (A10)$$

in the form of an integro-differential equation.

In an analogous manner, we derive the equation of transition

$$\partial_{\tau} \bar{P} = \bar{u} \frac{\partial \bar{P}}{\partial \lambda} + \frac{\partial}{\partial \lambda} \cdot K^{(0)}_{\lambda} \cdot \frac{\partial \bar{P}}{\partial \lambda} \quad (A11)$$

Here the governing diffusivity is

$$K^{(0)}_{\lambda} = (1 - A') K_{\lambda} \quad (A12)$$

as caused by the velocity fluctuations except \tilde{u}' .

We make a Fourier transformation of (A10) and (A11) by the formulas

$$\begin{aligned}
 (2\pi)^d P_0(-\tau, \underline{\kappa}) &= \int d\underline{\ell} e^{-i\underline{\kappa} \cdot \underline{\ell}} P_0(-\tau, \underline{\ell}) \\
 &= A_0 e^{-i\underline{\kappa} \cdot \underline{\ell}^{(0)}(-\tau)} \\
 (2\pi)^d \bar{P}(-\tau, \underline{\kappa}) &= \bar{A} e^{-i\underline{\kappa} \cdot \underline{\ell}^{(0)}(-\tau)},
 \end{aligned} \tag{A13}$$

where $\underline{\kappa}$ is an independent variable in the Fourier space, and d is the number of dimensions. We get

$$\partial_\tau P_0(-\tau, \underline{\kappa}) = \left[i\underline{\kappa} \cdot \underline{u}_0(-\tau) - \underline{\kappa} \underline{\kappa} : \underline{K}'_\ell \right] P_0(-\tau, \underline{\kappa}) \tag{A14a}$$

$$\partial_\tau \bar{P}(-\tau, \underline{\kappa}) = \left[i\underline{\kappa} \cdot \bar{\underline{u}} - \underline{\kappa} \underline{\kappa} : \underline{K}^{(0)}_\ell \right] \bar{P}(-\tau, \underline{\kappa}). \tag{A14b}$$

The solutions are:

$$\begin{aligned}
 (2\pi)^d P_0(-\tau, \underline{\kappa}) &= e^{-i\underline{\kappa} \cdot \underline{\ell}_0^{(0)}(-\tau)} \exp\left(-\underline{\kappa} \underline{\kappa} : \underline{K}'_\ell \tau\right) \\
 &= e^{-i\underline{\kappa} \cdot \underline{\ell}_0^{(0)}(-\tau)} h'(-\tau, \underline{\kappa}),
 \end{aligned} \tag{A15a}$$

$$\begin{aligned}
 (2\pi)^d \bar{P}(-\tau, \underline{\kappa}) &= \bar{h}(-\tau, \underline{\kappa}) \exp\left(-\underline{\kappa} \underline{\kappa} : \int_0^\tau d\tau' \underline{K}^{(0)}_\ell\right) \\
 &= \bar{h}(-\tau, \underline{\kappa}) \exp\left(-\frac{1}{2} \underline{\kappa} \underline{\kappa} : \left\langle \underline{u}^{(0)}_\tau \underline{u}^{(0)}_\tau \right\rangle_k \tau^2\right) \\
 &= \bar{h}(-\tau, \underline{\kappa}) h^{(0)}(-\tau, \underline{\kappa}),
 \end{aligned} \tag{A15b}$$

with the following components of the orbit function:

$$\bar{h}(-\tau, \kappa) \equiv \exp(-i\kappa \cdot \bar{u}\tau) \quad (\text{A16a})$$

$$h^{(0)}(-\tau, \kappa) \equiv \exp\left(-\frac{1}{2} \kappa \kappa : \left\langle \underset{\sim}{u}^{(0)} \underset{\sim}{u}^{(0)} \right\rangle \right|_{\kappa} \tau^2) \quad (\text{A16b})$$

$$h'(-\tau, \kappa) \equiv \exp\left(-\kappa \kappa : \underset{\sim}{K}_\ell' \tau\right). \quad (\text{A16c})$$

Note that $\underset{\sim}{K}_\ell'$ is the diffusivity in a matured (i.e. large-time τ) dispersion, and $\underset{\sim}{K}_\ell^{(0)}$ is the diffusivity in an unmatured (i.e. small-time τ) dispersion. In the latter case the orbit belongs to a free flight, so that

$$\chi \left\langle \underset{\sim}{u}^{(0)}(\kappa') \underset{\sim}{u}^{(0)}(-\kappa') \right\rangle \cong \chi \left\langle \underset{\sim}{u}^{(0)}(\kappa) \underset{\sim}{u}^{(0)}(\kappa) \right\rangle \quad (\text{A17})$$

is locally stationary, and the trace in isotropic turbulence becomes

$$\begin{aligned} \text{tr} \left\langle \underset{\sim}{u}^{(0)} \underset{\sim}{u}^{(0)} \right\rangle \Big|_{\kappa} &= \text{tr} \int d\kappa' \chi \left\langle \underset{\sim}{u}^{(0)}(\kappa') \underset{\sim}{u}^{(0)}(-\kappa') \right\rangle \\ &\cong \text{tr} \chi \left\langle \underset{\sim}{u}^{(0)}(\kappa) \underset{\sim}{u}^{(0)}(-\kappa) \right\rangle \int d\kappa', \end{aligned} \quad (\text{A18})$$

the integration over a region of radius κ gives

$$\begin{aligned} \int d\kappa' &= \pi \kappa^2 \quad \text{in 2 dimensions} \\ &= \frac{4\pi}{3} \kappa^3 \quad \text{in 3 dimensions} \end{aligned}$$

Here χ is the factor of Fourier truncation. In terms of the spectral function $F_u(\kappa)$, such that

$$2\pi\kappa \, \overline{t_r} \chi \left\langle \underset{\sim}{u^{(o)}}(\kappa) \underset{\sim}{u^{(o)}}(-\kappa) \right\rangle = F_u(\kappa) \quad \text{in 2 dimensions (A19a)}$$

$$4\pi \kappa^2 \, \overline{t_r} \chi \left\langle \underset{\sim}{u^{(o)}}(\kappa) \underset{\sim}{u^{(o)}}(-\kappa) \right\rangle = \frac{2}{3} F_u(\kappa) \quad \text{in 3 dimensions (A19b)}$$

we find

$$\overline{t_r} \left\langle \underset{\sim}{u^{(o)}} \underset{\sim}{u^{(o)}} \right\rangle \Big|_{\kappa} = \frac{2}{d^2} \kappa F_u(\kappa) \quad \text{in d dimensions (A20)}$$

In conclusion, with the use of (A20) and the introduction of the frequencies $m^{(o)}$ and ω'_D , such that

$$m^{(o)2} = \frac{1}{2} \kappa \kappa : \left\langle \underset{\sim}{u^{(o)}} \underset{\sim}{u^{(o)}} \right\rangle \Big|_{\kappa} = d^{-2} \kappa^3 F_u(\kappa) \quad (\text{A21a})$$

and

$$\omega'_K = \kappa \kappa : \underset{\sim}{K}' \approx \kappa^2 K', \quad (\text{A21b})$$

we find the orbit components to be

$$h^{(0)}(-\tau, \kappa) = \exp(-m^{(0)2} \tau^2) \quad (\text{A22a})$$

$$h'(-\tau, \kappa) = \exp(-\omega'_\kappa \tau) . \quad (\text{A22b})$$

Now the solutions (A15) can be applied to calculate

$$A_0 e^{-i\kappa \cdot \tilde{\ell}(-\tau)}$$

and, subsequently

$$\bar{A} A_0 e^{-i\kappa \cdot \tilde{\ell}(-\tau)}$$

by the use of the formulas (A13). We obtain

$$A_0 e^{-i\kappa \cdot \tilde{\ell}(-\tau)} = h'(-\tau, \kappa) e^{-i\kappa \cdot \tilde{\ell}_0(-\tau)} \quad (\text{A23a})$$

$$\bar{A} e^{-i\kappa \cdot \tilde{\ell}_0(-\tau)} = \bar{h}(-\tau, \kappa) h^{(0)}(-\tau, \kappa) . \quad (\text{A23b})$$

and hence

$$\bar{A} A_0 e^{-i\kappa \cdot \tilde{\ell}(-\tau)} = \bar{h}(-\tau, \kappa) h^{(0)}(-\tau, \kappa) h'(-\tau, \kappa) . \quad (\text{A24})$$

ACKNOWLEDGEMENT

This work was supported by the National Aeronautics and Space Administration.

REFERENCES

1. C.M. Tchen, "Kinetic approach to incompressible turbulence", C.R. Acad. Sci. Paris 286A, 605-607, (1978); "Kinetic approach to compressible turbulence", Proc. Int. Symposium on Rarefied Gas Dynamics, Cannes. Published by Commissariat à l'Energie Atomique, Paris, 1979, ed. by R. Campargue, pp. 105-114.
2. J.G. Charney, "Geostrophic turbulence". J. Atmos. Sci. 28, 1087-1095 (1971).
3. R. Fjørtoft, "On the changes in the spectral distribution of kinetic energy for two-dimensional non-divergent flow". Tellus 5, 225-230 (1953).
4. D.K. Lilly, "Numerical simulation of two-dimensional turbulence". Phys. Fluids, Suppl. II, 240-249 (1969).
5. D.K. Lilly, "Numerical simulation of two-dimensional turbulence". J. Fluid Mech. 45, part 2, 395-415 (1969).
6. P.D. Thompson, "The equilibrium energy spectrum of randomly forced two-dimensional turbulence". J. Atmos. Sci. 30, 1593-1598 (1973).
7. K.S. Gage, "Evidence for a $k^{-5/3}$ law inertial range in mesoscale two-dimensional turbulence". J. Atmos. Sci. 36, 1950-1954 (1979).
8. Ye.A. Novikov, "Spectral inequalities for two-dimensional turbulence". Izvestiya, Atmospheric and Oceanic Physics 14, 474-476 (1978).
9. C.E. Leith, "Diffusion approximation for two-dimensional turbulence". Phys. Fluids 11, 671-673 (1968).

10. P.G. Saffman, "A note on the spectrum and decay of random two-dimensional vorticity distributions at large Reynolds number". *Studies in Applied Mathematics*, M.I.T. Press 50, no. 4, 377-383 (1971).
11. P.B. Rhines, "Geostrophic turbulence". *Annual Review of Fluid Mechanics*, vol. 11, Annual Reviews, Inc., 401-441 (1979).
12. R.H. Kraichnan, "Inertia ranges in two-dimensional turbulence". *Phys. Rev.* 10, 1417-1423 (1967).
13. R.H. Kraichnan, "An almost-Markovian Galilean-invariant turbulence model". *J. Fluid Mech.* 47, 513-524 (1971).
14. R.H. Kraichnan, "Inertia range transfer in two- and three-dimensional turbulence". *J. Fluid Mech.* 47, 525-535 (1971).
15. J.R. Herring, "Statistical theory of quasi-geostrophic turbulence". *J. Atmos. Sci.* 37, 969-977 (1980).
16. A. Pouquet, M. Lesieur and J.C. André, "High Reynolds number simulation of two-dimensional turbulence using a stochastic model". *J. Fluid Mech.* 72, 305-319 (1975).
17. R. Salmon, G. Holloway and M.C. Hendershott, "The equilibrium statistical mechanics of simple quasi-geostrophic models". *J. Fluid Mech.* 75, 691-703 (1978).
18. C. Basdevant, M. Lesieur and R. Sadourny, "Subgrid-scale modeling of enstrophy transfer in two-dimensional turbulence". *J. Atmos. Sci.* 35, 1028-1042 (1978).
19. J. Weinstock, "On the theory of turbulence in the buoyancy subrange of stably stratified flows". *J. Atmos. Sci.* 35, 634-649 (1978).
20. W. Heisenberg, "Zur statistische theorie der turbulenz". *Zeit. Physik* 124, 628-657 (1948).
21. F.A. Gisina, "Spectral characteristics of turbulence in a thermally stratified atmosphere". *Izv. Atmos. Oceanic Phys.* 5, no. 3, 247-257 (1969). [English translation: *Bull. (Izv.) Acad. Sci. USSR, Atmos. Oceanic Phys.* 5, 137-141 (1969).
22. C.M. Tchen, "Repeated cascade theory of homogeneous turbulence". *Phys. Fluids* 16, 13-30 (1973).
23. C.M. Tchen, "Some mathematical relations which play a role in diffusion problems". *Verslagen Ned. Akad. Wetensch. Afd. Natuurkunde* 53, 400-410 (1944).

CHAPTER D: QUARTER FOUR

A Group-Kinetic Theory of Turbulent Collective Collision

C. M. Tchen

The City College and The Graduate School of The City University
of New York, N. Y. 10031, U.S.A.

AND J. H. Misguich

Association Euratom-CEA, Département de Physique du Plasma
Centre d'Etudes Nucléaires, 92260 Fontenay-aux-Roses, France

ABSTRACT

For the description of the microdynamical state of turbulence, a Liouville equation is taken that is equivalent to the basic hydrodynamical system of equations. The equation has the advantage of being homogeneous and contains less nonlinear terms. Our main objective is the derivation of the kinetic equation of turbulence which has a memory in the turbulent collision integral. We consider the basic pair-interaction, and the interaction between a fluctuation and the organized cluster of other fluctuations in the collection system, called the multiple interaction. By a group-scaling procedure, a fluctuation is decomposed into three groups to represent the three coupled transport processes of evolution, transport coefficient, and relaxation. By exploiting the property of quasi-stationarity at the different levels of degradation of coherence of the groups, we develop a transport theory with the closure by the memory-loss. The kinetic equation of the scaled singlet distribution is capable of investigating the spectrum of turbulence without the need of the knowledge of the pair distribution.

The exact propagator describes the detailed trajectory in the phase space, and is fundamental to the Lagrangian-Eulerian transformation. We calculate the propagator and its scaled groups by means of a probability of retrograde transition. Thus our derivation of the kinetic equation of the distribution involves a parallel development of the kinetic equations of the propagator and the transition probability. In this way, we can avoid the

assumptions of independence and normality.

Our result shows that the multiple interaction contributes to a shielding and an enhancement of the collision in weak turbulence and strong turbulence, respectively. The weak turbulence is dominated by the wave resonance, and the strong turbulence is dominated by the diffusion.

I. INTRODUCTION

The microdynamical state of turbulence can be described by the equation of evolution of the distribution function $f(t, \underline{x}, \underline{v})$ in the form:

$$(\partial_t + \hat{L}) \hat{f} = 0, \quad \partial_t \equiv \partial/\partial t. \quad (1a)$$

The differential operator

$$\hat{L} \equiv \underline{v} \cdot \underline{\nabla} + \hat{\underline{E}} \cdot \underline{\partial}, \quad \underline{\partial} \equiv \partial/\partial \underline{v} \quad (1b)$$

contains a fluctuating self-consistent field $\hat{\underline{E}}(t, \underline{x})$ which perturbs the ballistic orbit that is represented by $\partial_t + \underline{v} \cdot \underline{\nabla}$ in the phase space of position \underline{x} and velocity \underline{v} .

In plasma turbulence, (1a) is called the Vlasov equation, when $\hat{\underline{E}}$ is the electrostatic field multiplied by the ratio of electric charge to mass. The equation can also be used to treat the fluid turbulence, when $\hat{\underline{E}}$ represents the pressure-gradient or other additional hydrodynamical forces.

We decompose the total distribution function

$$\hat{f} = \bar{f} + \tilde{f} \quad (2)$$

into an average distribution

$$\bar{f} \equiv \langle f \rangle, \quad (3a)$$

by means of the averaging operator \bar{A} , and a fluctuation

$$\tilde{f} = \tilde{A} f, \quad (3b)$$

by means of the fluctuation operator

$$\tilde{A} = 1 - \bar{A}.$$

The difference between \bar{A} and $\langle \rangle$ lies in that the operator \bar{A} , like any other operators, applies to all the functions which may follow, while $\langle \rangle$ is an ensemble average of the function or functions limited by the angular brackets.

Thus by applying \bar{A} to (1a), i.e. by premultiplying (1a) by \bar{A} , we obtain the equation of evolution of \bar{f} , as follows:

$$(\partial_t + \bar{L})\bar{f} = \bar{C} . \quad (4)$$

The collective collision

$$\bar{C} = -\bar{A} \tilde{L} \tilde{f} , \quad (5)$$

can be written in the form

$$\bar{C} = \langle \tilde{\Delta} \rangle \{ \tilde{f} \} . \quad (6)$$

The collision operator

$$\langle \tilde{\Delta} \rangle = \partial_{\tilde{z}} \cdot \langle \tilde{D} \rangle \cdot \partial_{\tilde{z}} \quad (7)$$

has a diffusivity $\langle \tilde{D} \rangle$ which may serve as an operator to keep a memory.

On the other hand, by applying \tilde{A} to (1a), we obtain the equation of evolution of \tilde{f} . This equation may take the following alternative forms:

$$(\partial_t + \tilde{A} \hat{L})\tilde{f} = -\tilde{L} \tilde{f} \quad (8a)$$

$$(\partial_t + \bar{L})\tilde{f} = -\tilde{L} \tilde{f} + \tilde{C} \quad (8b)$$

$$(\partial_t + \hat{L})\tilde{f} = -\tilde{L} \tilde{f} - \bar{C} , \quad (8c)$$

where \bar{C} is defined by (5), and

$$\tilde{C} = -\tilde{A} \tilde{L} \tilde{f} = -\tilde{L} \tilde{f} - \bar{C} . \quad (9)$$

Our problem of the statistical theory of turbulence is to find the kinetic equation (4), with the turbulent collision (5) to be derived on the basis of one of the three equations (8a) - (8c) of fluctuations. The three equations differ by their governing differentials

$$\partial_t + \tilde{A} \hat{L}, \quad \partial_t + \bar{L}, \quad \partial_t + \hat{L}, \quad (10)$$

so that their integrations will have different evolution operators, as follows:

$$\hat{\Lambda}, \quad U_{free}, \quad \hat{U}. \quad (11)$$

The operator $\hat{\Lambda}$ was introduced by Weinstock,¹⁻³ U_{free} is the free flight or ballistic propagator, describing a trajectory that is not perturbed by the \hat{E} - field fluctuations, and \hat{U} is the exact propagator.

The differential equation (8a) has the simplest source, but its operator $\hat{\Lambda}$ is complicate. On the other hand, the integration of the differential equation (8b) can be achieved by the simplest propagator U_{free} of free flight, but the source is complicated by the nonlinear fluctuation \tilde{C} , as this will generate a hierarchy and requires a closure.⁴ In the following we choose (8c), because the average source \bar{C} is easier than the random source \tilde{C} , and the propagator \hat{U} is simpler than the operator $\hat{\Lambda}$, and is governed by the well determined equation of the exact trajectory in the form:

$$[\partial_t + \hat{L}(t)] \hat{U}(t, t') = 0, \quad (12a)$$

or

$$[\partial_{t'} + \hat{L}(t')] \hat{U}(t, t') = 0. \quad (12b)$$

The fluctuating source $\tilde{L} \bar{f}$ gives a pair collision:⁵

$$\bar{C} = \langle \tilde{A} \rangle \{ \bar{f} \}, \quad (13a)$$

that is of the diffusive type with an operator

$$\langle \tilde{A} \rangle = \partial_{\tilde{x}} \cdot \langle \tilde{D} \rangle \cdot \partial_{\tilde{x}} \quad (13b)$$

related to the diffusivity $\langle \tilde{D} \rangle$. The two diffusivities are defined by:

$$\langle \tilde{D} \rangle = \int_0^t d\tau \langle \tilde{E}(t, \tilde{x}) \hat{\Lambda}(t, t-\tau) \tilde{E}(t-\tau) \rangle \quad (14a)$$

$$\langle \tilde{D} \rangle = \int_0^t d\tau \langle \tilde{E}(t, \tilde{x}) \hat{U}(t, t-\tau) \tilde{E}(t-\tau) \rangle. \quad (14b)$$

Different levels of approximations have been made in the literature on the theories of turbulence in fluids and plasmas. As the lowest order, the quasi-linear theory assumed a propagator of free flight U_{free} and obtained a resonance denominator $\omega - \underline{k} \cdot \underline{v}$, where ω and \underline{k} are the frequency and the wavenumber of a wave

$$e^{-i(\omega - \underline{k} \cdot \underline{v})\tau} \quad (15)$$

interacting with the particle of velocity \underline{v} . This resonance was fundamental to the Landau damping in the quasilinear theory. The next order approximation assumed an expansion around \bar{U} in the renormalization theory. Interesting advances in theory of turbulence have made use of this assumption.^{6,7}

With the group scaling, the derivation of the spectrum of turbulence can be made on the basis of the kinetic equation of the singlet distribution without the pair distribution.⁸ We represent the three transport processes of evolution, transport coefficient, and relaxation, by three scaled groups and analyse their interactions. The closure is obtained by the loss of memory in the processes and by exploiting the property of quasi-stationarity in the degrading coherence of groups.

The paper is organized as follows. After a comparison between the hydrodynamical and the kinetic descriptions of the microdynamical state of turbulence (Sec. II), we introduce a group-scaling procedure for representing the three transport processes by the three groups (Sec. III). We find that the kinetic equation of the macro-distribution has a turbulent collision integral with a memory. We investigate the pair interaction and the multiple interaction between a micro-fluctuation and the cluster of other micro-fluctuations that form a macro-group. Various forms of memory may develop:

(i) First, the memory appears in the collision that is controlled by the diffusivity acting as an integral operator. Then, since the collective collision includes the pair collision and the multiple collision in the form of an integral equation, a memory ensues.

(ii) The Lagrangian representation of a function is made by means of a propagator that is governed by a kinetic equation with a memory.

Obviously, the memory group (i) which deals with the evolution of the distribution function at the time instant t has a longer duration than the memory group (ii) which deals with the evolution of the propagator

for the shorter time-interval $t-t'$. We shall neglect the memory of the group (ii), by justifying from the property of quasi-stationarity between the groups (Sec. IV). We shall show that the propagator is related to the probability of retrograde transition, and develop two parallel kinetic theories on the basis of the detailed dynamics of the trajectory (Sec. V). Finally, the collective collision is investigated, by means of a closure based upon the memory-loss (Sec. VI). A memory function appears and is controlled by the resonance function in weak turbulence and by the diffusion in strong turbulence, yielding two opposite outcomes: a shielded collision in weak turbulence, and an enhanced collision in strong turbulence (Sec. VII).

II. MICRODYNAMICAL STATE OF TURBULENCE.

The microdynamical state of turbulence can be described in two ways. First we can use the Navier-Stokes equations for the incompressible fluid:

$$\left(\frac{\partial}{\partial t} + \hat{\underline{u}} \cdot \nabla - \nu \nabla^2 \right) \hat{\underline{u}} = \hat{\underline{E}} \quad (16a)$$

$$\nabla \cdot \hat{\underline{u}} = 0 \quad (16b)$$

Here $\hat{\underline{u}}$ is the fluid velocity, ν is the kinematic viscosity, and

$$\hat{\underline{E}} = -\frac{1}{\rho} \nabla \hat{p} \quad (17)$$

is the pressure gradient, and may even include other forces, such as buoyancy and random sources.

Alternatively, we can consider the microkinetic equation (1a),
rewritten in the form of the Liouville equation:

$$[\partial_t + \hat{L}(t)] \hat{f}(t, \underline{x}, \underline{v}) = 0. \quad (18)$$

The evolution of the exact distribution function $\hat{f}(t, \underline{x}, \underline{v})$ is controlled by the differential operator

$$\hat{L}(t) = \underline{v} \cdot \underline{\nabla} - \gamma \underline{\nabla}^2 \hat{E}(t, \underline{x}) \cdot \underline{\partial}. \quad (19)$$

The microkinetic equation has been used as a basis of deriving the BBGK equations by writing $\hat{f}(t, \underline{x}, \underline{v})$ in the form of a summation of δ -functions for N particles in the phase space.⁸ Presently, in order to secure the equivalence of the two descriptions, we write \hat{f} in the form of a single δ -function:

$$\hat{f}(t, \underline{x}, \underline{v}) = \delta[\underline{v} - \underline{u}(t, \underline{x})]. \quad (20)$$

The density in the incompressible fluid is taken as unity here without loss of generality. The compressible fluid should have the density $\rho(t, \underline{x})$ as a factor before the δ -function.

By taking the moments, it can be demonstrated that the microkinetic description is transformed into the fluid description as represented by the Navier-Stokes equations.

By the condition of incompressibility, we can write \hat{E} in the following two forms:

$$\hat{E} = -\frac{1}{4\pi} \underline{\nabla} \cdot \int d\underline{x}' \frac{1}{|\underline{x} - \underline{x}'|} \underline{\nabla}' \underline{\nabla}' : \hat{u}(t, \underline{x}') \hat{u}(t, \underline{x}'), \quad \underline{\nabla}' \equiv \partial/\partial \underline{x}', \quad (20)$$

$$\hat{\tilde{E}} = -\frac{1}{4\pi} \nabla \cdot \int d\mathbf{x}' d\mathbf{v}' \frac{1}{|\mathbf{x}-\mathbf{x}'|} (\mathbf{v}' \cdot \nabla')^2 \hat{f}(t, \mathbf{x}', \mathbf{v}'), \quad (21)$$

One is nonlinear and the other is linear, so that the differential operator becomes

$$\tilde{L} = \mathcal{E} \{ \tilde{f} \} \quad , \quad (22a)$$

with

$$\mathcal{E} = -\frac{1}{4\pi} \nabla \cdot \partial \int d\mathbf{x}' d\mathbf{v}' \frac{1}{|\mathbf{x}-\mathbf{x}'|} (\mathbf{v}' \cdot \nabla')^2 . \quad (22b)$$

The Navier-Stokes equations (16) and the Liouville equation (18) may be called the primitive equations in the physical and phase spaces, respectively.

The kinetic approach has several advantages. First, it transforms the inhomogeneous Navier-Stokes equation into the homogeneous equation (18) and rendered all nonlinear terms, as arising from the velocity function $\hat{u}(t, \mathbf{x})$, into linear ones with the independent variable \mathbf{v} , as in (18) and (21). The only nonlinear term kept is connected with $\hat{\tilde{E}}$ in (18) for describing the mode-couplings. It can ultimately be treated by the linear equation of state (21).

Secondly, the kinetic approach treats both the wave-wave interaction and the wave-particle interaction. The latter is the mechanism of the linear and nonlinear Landau dampings or amplifications. This interaction is not explicit in the hydrodynamical approach.

Finally, should one propose a general kinetic theory that combines both the molecular motion and the turbulent motion, the microkinetic equation (18) can again serve as a basis, provided $\hat{\tilde{E}}$ includes both motions above, without a predetermined viscosity. Our kinetic theory that is based upon (18) with a differential operator (19) is concerned with the turbulent motion in a viscous fluid medium. However, for the investigation of the turbulent collisions by eddies of size larger than the viscous cutoff, the effect of ν can be neglected.

III. GROUP-SCALING PROCEDURE

The decomposition of a fluctuating function into an average and a fluctuation is the usual practice in statistical methods. We denote the global average and the fluctuation by

$$\bar{A} \quad \text{and} \quad \tilde{A} = 1 - \bar{A} \quad , \quad (23)$$

respectively.

Not all scales of fluctuations perform the same role in the three processes mentioned earlier: evolution, transport coefficient, and relaxation. For describing these processes and their couplings analytically, we represent them by three scales, using the three operators

$$A^{(0)}, \quad A' \quad , \quad A'' \quad , \quad (24a)$$

to form the three groups, or scaled fluctuations:

$$\tilde{E}^0 = A^{(0)} \hat{\tilde{E}} \quad , \quad \tilde{E}' = A' \hat{\tilde{E}} \quad , \quad \tilde{E}'' = A'' \hat{\tilde{E}} \quad . \quad (24b)$$

They fluctuate and have their durations of correlation

$$\tau_c^{\circ} > \tau_c' > \tau_c'' \quad (25)$$

in the increasing degrees of incoherence. This is the property of degradation of group coherence, instituting a quasi-stationarity of one group with respect to the other. The fields (24b) will also be called macro-, micro- and submicro-fields, respectively.

The two-scale averaging procedure is obtained by further scaling \tilde{A} into

$$\tilde{A} = A^{\circ} + A' \quad , \quad (26)$$

by using $A^{(0)}$ to find

$$A^{\circ} \bar{A} = 0, \quad A^{\circ} A^{\circ} = A^{\circ}, \quad A^{\circ} A' = 0.$$

Subsequently, the three-scale averaging procedure is obtained by another scaling of A' into

$$A' = A^{(1)} + A'' \quad , \quad (27)$$

and using $A^{(1)}$ to find

$$A^{(1)} \bar{A} = 0, \quad A^{(1)} A^{\circ} = 0, \quad A^{(1)} A^{(1)} = A^{(1)}, \quad A^{(1)} A'' = 0.$$

The averaging procedures of many scales may be denoted by

$$\bar{A} \equiv \langle \rangle, \quad A_0 \equiv \langle \rangle_0, \quad A_1 \equiv \langle \rangle_1, \quad (28a)$$

where \bar{A} is the operator of the global average, A_0 and A_1 are the operators of the accumulated averages:

$$A_0 = \bar{A} + A^{\circ}, \quad A_1 = \bar{A} + A^{\circ} + A^{(1)}. \quad (28b)$$

Although the groups A^0 and A' may overlap in their wavenumbers instantaneously, their statistical functions, e.g. their spectral contents must lie in adjacent domains of wavenumbers, i.e.

$$(0,k) \quad , \quad (k,\infty) \quad . \quad (29)$$

We conclude that the group-scaling procedure is a scaling into the three transport processes mentioned above. It can distinguish between the two diffusivities: the asymptotic diffusivity $\langle D' \rangle$ and the non-asymptotic diffusivity $\langle \underline{D}^0 \rangle$, a distinction not present in the one-scale averaging procedure.

Recall that the spectral function, being the Fourier transform of the correlation function of velocities at two points, would, in the one-scale averaging, require a pair distribution function and its coupling with the singlet distribution function. In the present procedure of group-scaling, the singlet distribution function f^0 suffices. Indeed, f^0 gives \underline{u}^0 and $\langle (\underline{u}^0)^2 \rangle$, and thus derives the spectrum of \underline{u} -fluctuations by a differentiation with respect to k .

The degradation of group coherence (25) indicates a consecutive quasi-stationarity among the groups, and is an important property for classifying the interactions and placing the memory and the memory-loss.

IV. TURBULENT COLLISIONS AND MEMORIES

A. Collision in the Kinetic Equation of f_o

By scaling the Liouville equation (18) by means of A_o and A' , we get the equations of evolution of the macro-distribution

$$(\partial_t + A_o L_o) f_o = C_o \quad (30)$$

and the micro-distribution in the form:

$$(\partial_t + \hat{L}) f' = -L' f_o - C_o \quad (31a)$$

or, equivalently,

$$(\partial_t + A' \hat{L}) f' = -L' f_o \quad (31b)$$

Here

$$C_o = -A_o L' f' \quad (32)$$

is the collective collision. The scheme falls into the framework that was described by (8c) and (8a).

The differential equation (31a) can be integrated to give

$$\begin{aligned} f' = & -A' \int_0^t dt' \hat{U}(t, t') L'(t') f_o(t') \\ & -A' \int_0^t dt' \hat{U}(t, t') C_o(t') \end{aligned} \quad (33)$$

The lower limit of integration is set to zero, and the initial value is ignored, since

$$A_0 L'(t) f'(0) = 0 \quad . \quad (34)$$

Here and in the following, the dependence on \underline{x} or $(\underline{x}, \underline{v})$ is understood.

The propagator

$$\hat{U}(t, t') = \exp \left[- \int_{t'}^t dt'' \hat{L}(t'') \right]$$

satisfies the differential equations (12).

It is to be remarked that the differential equation (31a) for the evolution of f' has two sources written in the right hand side: The micro-source $-L'f_0$ represents a coupling between the micro-operator L' and the macro-distribution f_0 and gives a fluctuating contribution to the solution, as seen from the first term in the right hand side of (33). The macro-source C_0 represents the collisional cluster in the medium in which f_0 evolves. Although C_0 is quasi-stationary, it can give a random contribution through the operation $A'\hat{U}$ as seen from the second term in the right hand side. This term represents the effect of the quasi-stationary C_0 riding on the random trajectory.

In order to determine the collective collision, as defined by (32), we multiply (33) by $L'(t)$ and average, obtaining:

$$C_0(t) = A_0 \int_0^t dt' L'(t) \hat{U}(t, t') L'(t') f_0(t')$$

$$+ A_0 \int_0^t dt' L'(t) U'(t, t') C_0(t') . \quad (36)$$

This expression consists of the pair collision

$$\phi^0(t) = A_0 \int_0^t dt' L'(t) U(t, t') L'(t') f_0(t') , \quad (37)$$

and the multiple collision in the form:

$$\begin{aligned} A_0 \int_0^t dt' L'(t) U'(t, t') C_0(t') &= - A_0 \int_0^t dt' H_0(t, t') C_0(t') \\ &= - A_0 H_0 * C_0 , \end{aligned} \quad (38)$$

Here H_0 can be identified as the collective collision in the kinetic equation of U_0 , i.e.

$$\left[\frac{\partial}{\partial t} + L_0(t) \right] U_0(t, t') = - A_0 L'(t) U'(t, t') \equiv H_0(t, t') , \quad (39)$$

as obtained upon scaling (12a) by A_0 . Note that (12b) will not lead to the desired collision and is therefore not relevant here.

As an option to (8a), we have the equation of evolution of the micro-distribution in the form

$$\left(\frac{\partial}{\partial t} + A' \hat{L} \right) f' = - L' f_0 , \quad (40)$$

and obtain

$$f' = - A' \int_0^t dt' \hat{\Lambda}(t, t') L'(t') f_0(t') \quad (41)$$

by an integration and the use of the operator $\hat{\Lambda}$. It follows:

$$\begin{aligned}
 C_o &= - A_o L' f' \\
 &= A_o \int_0^t dt' L'(t) \hat{\Lambda}(t, t') L'(t') f_o(t') \\
 &= A_o \Delta' \{f_o\} \quad , \quad (42)
 \end{aligned}$$

Here $A_o \Delta'$ denotes the collision operator

$$\begin{aligned}
 A_o \Delta' &= A_o \int_0^t dt' L'(t) \hat{\Lambda}(t, t') L'(t') \\
 &= \partial \cdot A_o \mathcal{D}' \cdot \partial \quad , \quad (43)
 \end{aligned}$$

with a diffusivity

$$A_o \mathcal{D}' = A_o \int_0^t dt' \tilde{E}'(t) \hat{\Lambda}(t, t') \tilde{E}'(t') \quad . \quad (44)$$

By replacing $\hat{\Lambda}$ by \hat{U} in (42) - (44), we obtain the pair collision

$$C_o = A_o \Delta' \{f_o\} \quad , \quad (45)$$

with a collision operator

$$\begin{aligned}
 A_o \Delta' &= A_o \int_0^t dt' L'(t) \hat{U}(t, t') L'(t') \\
 &\equiv \partial \cdot A_o \mathcal{D}' \cdot \partial \quad , \quad (46)
 \end{aligned}$$

and a diffusivity

$$A_0 \underline{D}' = A_0 \int_0^t dt' \underline{E}'(t) \hat{U}(t, t') \underline{E}'(t') \quad (47)$$

By collecting the results (36) - (38), we obtain the following relation between the two collisions C_0 and \mathbb{C}_0 in the form:

$$C_0 = \mathbb{C}_0 - A_0 H_0 * C_0, \quad (48)$$

with

$$A_0 H_0 * C_0 = A_0 \int_0^t dt' H_0(t, t') C_0(t') \quad (49)$$

B. Collision in the Kinetic Equation of U_0

The distribution function \hat{f} and the propagator \hat{U} have their evolutions of some resemblance. By repeating the calculations that have been made in Subsec. IVA, we obtain the collisions

$$H_0 = A_0 \Delta' \{ U_0 \} \quad (50a)$$

$$\mathbb{H}_0 = A_0 \Delta' \{ U_0 \} \quad (50b)$$

of U_0 , as related by the formula:

$$H_0 = \mathbb{H}_0 - H_0 * H_0 \quad (51)$$

Here H_0 and \mathbb{H}_0 are the collective collision and the pair collision in the equation of evolution of U_0 , i.e. (39). The results (50a), (50b), and (51) are analogous to (42), (45) and (48), respectively.

C. Memories

We summarise the results obtained above as follows:

(i) Primary memory in C_0

The kinetic equation of f_0 , in the form (30) and with the collision (42), carries a primary memory by the collision operator $A_0 \Delta' \{ \}$, which can be written into two groups as follows:

$$A_0 \Delta' \{ \} = \langle \Delta' \rangle \{ \} + A_0 (\Delta')^0 \{ \} \quad (52a)$$

The second group forms a correlation $A_0 (\Delta')^0 \{ f_0 \}$ on its own, but does not have sufficient time to approach its equilibrium, by (25), and therefore will not contribute to a diffusive collision. Hence we can write

$$A_0 \Delta' \{ \} \approx \langle \Delta' \rangle A_0 \{ \} \quad (52b)$$

as an approximation, reducing (42) - (49) into:

$$C_0 = \langle \Delta' \rangle \{ f_0 \}, \quad \mathbb{C}_0 = \langle \Delta' \rangle \{ f_0 \} \quad (53a)$$

$$\bar{H} = \langle \Delta' \rangle \bar{U}, \quad \bar{H} = \langle \Delta' \rangle \bar{U} \quad (53b)$$

$$C_0 = \mathbb{C}_0 - \bar{H} * C_0 \quad (53c)$$

$$\bar{H} = \bar{H} - \bar{H} * \bar{H}, \quad (53d)$$

with

$$\langle \Delta' \rangle = \partial \cdot \langle \underline{D}' \rangle \cdot \partial, \quad \langle \Delta' \rangle = \partial \cdot \langle \underline{D}' \rangle \cdot \partial \quad (54)$$

$$\langle \underline{D}' \rangle = \int_0^t dt' \langle \underline{E}(t) \hat{\Lambda}(t, t') \underline{E}(t') \rangle, \quad \langle \underline{D}' \rangle = \int_0^t dt' \langle \underline{E}(t) \hat{U}(t, t') \underline{E}(t') \rangle$$

Hence the evolution

$$\begin{aligned}
 (\partial_t + A_0 L_0) f_0(t, \underline{x}, \underline{v}) &= \langle \Delta' \rangle \{ f_0(t') \} \\
 &= \int_0^t d\tau \langle L'(t) \hat{\Lambda}(t, t-\tau) L'(t-\tau) \rangle \{ f_0(t-\tau) \}, \quad (55)
 \end{aligned}$$

with $t-t' = \tau$, carries a primary memory in C_0 from the fluctuations E' of scale τ_c' , such that

$$t > \tau_c' > \tau, \quad (56)$$

by (25).

(ii) Secondary memory in H_0

On the other hand, if we write $\hat{U}(t, t') = \hat{U}(\tau)$, with $t-t' = \tau$, the kinetic equation

$$\begin{aligned}
 (\partial_t + A_0 L_0) U_0(t, t-\tau) &= H_0(t, t-\tau) \\
 &= \langle \Delta' \rangle \{ U_0(\tau-\tau) \} \\
 &= \int_0^\tau d\tau' \langle L'(\tau) \hat{\Lambda}(\tau, \tau-\tau') L'(\tau-\tau') \rangle U_0(\tau-\tau') \quad (57)
 \end{aligned}$$

has a collision integral governed by the collision operator

$$\begin{aligned}
 &\int_0^\tau d\tau' \langle L'(\tau) \hat{\Lambda}(\tau, \tau-\tau') L'(\tau-\tau') \rangle \\
 &= \int_0^\tau d\tau' \langle L^{(1)}(\tau) \hat{\Lambda}(\tau, \tau-\tau') L^{(1)}(\tau-\tau') \rangle \\
 &+ \int_0^\tau d\tau' \langle L^{(2)}(\tau) \hat{\Lambda}(\tau, \tau-\tau') L^{(2)}(\tau-\tau') \rangle
 \end{aligned}$$

decomposed into two parts. With the condition of quasi-stationarity

$$\tau > \tau_c'' > \tau' \quad (58)$$

as a continuation of (56) from (25), the second integral has a matured diffusivity, where we can put $\tau \rightarrow \infty$ and obtain $\langle \Delta''(\tau \rightarrow \infty) \rangle$, while the first integral has an unmatured diffusivity. Since $\tau' \ll \tau$, the secondary memory carried by the collision integral H_0 in the short time span τ can be assumed negligible as compared with the primary memory.

D. Resolution of the Integral Equations

From the definition

$$\bar{H}(\omega) = \frac{1}{\pi} \int_0^{\infty} d\tau \, e^{-i\omega\tau} \bar{H}(\tau), \quad (59)$$

the Fourier transform of the convolution $\bar{H} * C_0$ is

$$\begin{aligned} & \frac{1}{\pi} \int_0^{\infty} d\tau \, e^{-i\omega\tau} \int_0^{\tau} d\tau' \, \bar{H}(\tau') C_0(\tau - \tau') \\ &= \frac{1}{\pi} \int_0^{\infty} d\tau'' \int_0^{\infty} d\tau' \, e^{-i\omega\tau''} e^{-i\omega\tau'} \bar{H}(\tau') C_0(\tau''), \end{aligned}$$

and is written as

$$\bar{H} * C_0 \doteq \pi \bar{H}(\omega) C_0(\omega). \quad (60)$$

Here a change of variables $\tau'' = \tau - \tau'$ has been made. The symbol \doteq denotes a Fourier transformation. Hence we transform (53c) into:

$$C_0(\omega) = C_0(\omega) - \pi \bar{H}(\omega) C_0(\omega). \quad (61)$$

V. KINETIC EQUATION OF THE TRANSITION PROBABILITY

A. Path Dynamics

The trajectory with which the propagator evolves is described by the following differential equation of the path dynamics:

$$\frac{d^2 \hat{\underline{x}}(t')}{dt'^2} = \hat{\underline{E}}(t'), \quad (62)$$

or, identically, by the system

$$\frac{d \hat{\underline{x}}(t')}{dt'} = \hat{\underline{v}}(t'), \quad \frac{d \hat{\underline{v}}(t')}{dt'} = \hat{\underline{E}}(t') \quad (63)$$

with the conditions

$$\hat{\underline{x}}(t) = \underline{x}, \quad \hat{\underline{v}}(t) = \underline{v}, \quad (64)$$

which specify that at the instant of time t , the said trajectory passes by the phase point $(\underline{x}, \underline{v})$.

By integrating (63) and writing $t' = t - \tau$, we find the dynamical variables

$$\hat{\underline{x}}(t - \tau) = \underline{x} + \hat{\underline{\ell}}(-\tau), \quad \hat{\underline{v}}(t - \tau) = \underline{v} + \hat{\underline{V}}(-\tau) \quad (65)$$

and the displacements

$$\begin{aligned} \hat{\underline{\ell}}(-\tau) &\equiv - \int_0^\tau d\tau' \hat{\underline{v}}(t - \tau') \\ &= - \underline{v} \tau + \int_0^\tau d\tau' (\tau - \tau') \hat{\underline{E}}(t - \tau') \end{aligned} \quad (66a)$$

$$\hat{\underline{V}}(-\tau) = - \int_0^\tau d\tau' \hat{\underline{E}}(t - \tau'). \quad (66b)$$

Now the path dynamics (63) that is driven by $\hat{\underline{E}}$ has the displacements:

$$\frac{d\hat{\underline{l}}(-\tau)}{d\tau} = - \hat{\underline{v}}(t-\tau) \quad (67a)$$

$$\frac{d\hat{\underline{v}}(t-\tau)}{d\tau} = - \hat{\underline{E}}(t-\tau) . \quad (67b)$$

B. Retrograde Transition Probability

The Lagrangian correspondent of an Eulerian function $G(t, \underline{x})$ can be written in the form

$$G \left[t-\tau, \underline{x} + \hat{\underline{l}}(-\tau) \mid t, \underline{x} \right], \text{ with } \tau > 0, \quad (68a)$$

and specifies the value of the function as observed by a fluid particle coming from the point $\hat{\underline{x}}(t-\tau)$ at the time $t-\tau$, along the trajectory which at the time t passes the point \underline{x} . The Lagrangian function can be written more conveniently as

$$\hat{U}(t, t-\tau) G(t-\tau) \quad (68b)$$

by means of the propagator $\hat{U}(t, t-\tau)$. It is in reality a function of two states:

$$t-\tau, \underline{x} + \hat{\underline{l}}(-\tau) \quad \text{and} \quad t, \underline{x} . \quad (69)$$

Since $\hat{\underline{l}}(-\tau)$ is a random function, we can introduce a retrograde transition probability, written in the form:

$$\hat{p}(t-\tau, \underline{x} + \underline{l} | t, \underline{x}) , \quad (70a)$$

or, briefly in the notation

$$\hat{p}(-\tau, \underline{l}) . \quad (70b)$$

The first form expresses the state t, \underline{x} as a condition and can be called a conditional probability. The second form implies a quasi-stationary transition, i.e. \hat{p} varies more rapidly with $-\tau, \underline{l}$ than with t, \underline{x} . Here \underline{l} is an independent variable.

The concept of the retrograde transition probability has been introduced earlier and its equation of evolution was found to be of the Fokker-Planck type.⁹ More recently, by the use of this transition probability, the kinetic equation of turbulence was found.¹⁰ In the present Section, we shall devote to the relation between the retrograde transition probability and the propagator.

As the basis of the dynamics of $\hat{p}(-\tau, \underline{l})$, we write the equation of the detailed evolution in the form

$$\left(\frac{\partial}{\partial \tau} + \hat{L}(t-\tau) \right) \hat{p}(-\tau, \underline{l}) = 0 , \quad (71a)$$

satisfying the condition of normalization:

$$\int d\underline{l} \hat{p}(-\tau, \underline{l}) = 1 . \quad (71b)$$

The differential operator is

$$\hat{\mathcal{L}}(t-\tau) \equiv -\hat{\mathbf{v}}(t-\tau) \cdot \frac{\partial}{\partial \underline{\ell}} \quad (72)$$

In order for (71b) to describe the microdynamical state of the turbulent trajectory, we put

$$\hat{p}(-\tau, \underline{\ell}) = \delta[\underline{\ell} - \hat{\underline{\ell}}(-\tau)] \quad (73)$$

The partial differential equation (71a) has its characteristic equations coinciding with (67). Evidently, by substituting (73) and integrating (71a) with respect to $\underline{\ell}$, we will reproduce the basic dynamical equations (67).

The Fourier transforms of (71a) and (71b) are:

$$\left[\frac{\partial}{\partial \tau} + \hat{\mathcal{L}}(t-\tau, \underline{k}) \right] \hat{p}(-\tau, \underline{k}) = 0 \quad (74a)$$

and

$$\hat{p}(-\tau, \underline{k}=0) = (2\pi)^{-d}, \quad (74b)$$

with

$$\hat{\mathcal{L}}(t-\tau, \underline{k}) \equiv -i\underline{k} \cdot \hat{\mathbf{v}}(t-\tau) \quad (75a)$$

and

$$\int_0^\tau d\tau' \hat{\mathcal{L}}(t-\tau', \underline{k}) = -i\underline{k} \cdot \hat{\underline{\ell}}(-\tau) \quad (75b)$$

Here $d=3$ in the three dimensional space. The integration of (74a) with the condition (74b) gives the solution

$$\begin{aligned}
\hat{p}(-\tau, \underline{k}) &= (2\pi)^{-d} \exp \left[-\int_0^\tau d\tau' \hat{\mathcal{L}}(-\tau', \underline{k}) \right] \\
&= (2\pi)^{-d} \exp i \underline{k} \cdot \hat{\underline{l}}(-\tau) \\
&= (2\pi)^{-d} e^{-i \underline{k} \cdot \underline{v} \tau} \hat{p}(-\tau, \underline{k}).
\end{aligned} \tag{76a}$$

It consists of a ballistic orbit function $e^{-i \underline{k} \cdot \underline{v} \tau}$ and a field-dependent orbit function, which is

$$\hat{p}(-\tau, \underline{k}) = \exp i \underline{k} \cdot \tilde{\underline{l}}(-\tau), \tag{76b}$$

if $\underline{E} = 0$.

C. Relation Between the Propagator and the Transition Probability

The evolution of \bar{U} is governed by a kinetic equation with a collision \bar{H} . It is to be noted that $\hat{U}(t, t')$, being an evolution operator, or propagator, describes the exact trajectory, and therefore is a functional of the path $\hat{\underline{l}}(-\tau)$ during the interval of time from t to $t-\tau$ in the retrograde transition. Hence the propagator can be written as:

$$\hat{U}(t, t') = \hat{U}[\hat{\underline{l}}(-\tau)], \tag{77a}$$

or

$$\hat{U}(t, t') = \int d\underline{l} \hat{U}(\underline{l}) \hat{p}(-\tau, \underline{l}), \quad t-t' = \tau, \tag{77b}$$

by the use of the transition probability $\hat{p}(-\tau, \underline{l})$. In an analogous manner, we have:

$$\bar{U}(t, t') = \int d\underline{l} \bar{U}(\underline{l}) \bar{p}(-\tau, \underline{l}). \tag{77c}$$

In this way, the propagator is determined by the probability of transition

$$\hat{p}(-\tau, \underline{l}) , \text{ or } \bar{p}(-\tau, \underline{l}) .$$

D. Lagrangian-Eulerian Transformation

The Lagrangian function depends on two states, and can be written in three forms, by means of a functional as in (68a), a propagator as in (68b), and now a transition probability in the form:

$$\hat{U}(t, t-\tau) G(t-\tau) = \int d\underline{l} \hat{p}(-\tau, \underline{l}) G(t-\tau, \underline{x} + \underline{l}) . \quad (78a)$$

It admits a Fourier form:

$$\hat{U}(t, t-\tau) G(t-\tau) = (2\pi)^d \hat{p}(-\tau, \underline{k}) G(t-\tau, \underline{k}), \quad (78b)$$

by a Fourier transformation with respect to \underline{l} . A time integration yields the convolution:

$$\hat{U} * G = \int_0^t d\tau \hat{U}(t, t-\tau) G(t-\tau) . \quad (79a)$$

Hence the combined Fourier transformations with respect to both t and \underline{l} yield :

$$\hat{U} * G = (2\pi)^{-d} \pi \hat{p}(-\omega, \underline{k}) G(\omega, \underline{k}) \quad (79b)$$

or

$$\hat{U} * G = \pi \left[\int_0^\infty d\tau e^{i(\omega - \underline{k} \cdot \underline{v})\tau} \hat{p}(-\tau, \underline{k}) \right] G(\omega, \underline{k}) . \quad (79c)$$

Use of (76b) has been made.

We conclude that the transition probability helps in determining the propagator and the function to be operated upon, and thereby establishes the Lagrangian-Eulerian transformation.

E. Kinetic Equation

By scaling (71a) by \bar{A} and \tilde{A} , we find the system:

$$\left[\frac{\partial}{\partial \tau} + \tilde{\mathcal{L}}(t-\tau) \right] \bar{p}(-\tau, \underline{l}) = \bar{J}(-\tau, \underline{l}) \quad (80a)$$

$$\left[\frac{\partial}{\partial \tau} + \hat{\mathcal{L}}(t-\tau) \right] \tilde{p}(-\tau, \underline{l}) = -\tilde{\mathcal{L}}(t-\tau) \bar{p}(-\tau, \underline{l}) - \bar{J}(-\tau, \underline{l}), \quad (80b)$$

with the collision

$$\bar{J}(-\tau, \underline{l}) \equiv - \langle \tilde{\mathcal{L}}(t-\tau) \tilde{p}(-\tau, \underline{l}) \rangle. \quad (80c)$$

The procedure of determining the collision $\bar{J}(-\tau, \underline{l})$ is the same as that yielded the collision \bar{H} in the kinetic equation of \bar{U} . We deduce

$$\bar{J} = \bar{\mathcal{J}} - \bar{H} * \bar{J}, \quad (81)$$

analytically, as was with (53c) and (53d). We have

$$\bar{J}(-\tau, \underline{l}) = \underline{\nabla} \cdot \langle \tilde{K}(\tau) \rangle \cdot \underline{\nabla} \bar{p}(-\tau, \underline{l}) \quad (82a)$$

$$\bar{J}(-\tau, \underline{\ell}) = \nabla \cdot \langle \tilde{K}(\tau) \rangle \cdot \nabla \bar{p}(-\tau, \underline{\ell}) , \quad (82b)$$

with

$$\langle \tilde{K} \rangle = \int_0^\tau d\tau' \cdot \langle \tilde{V}(-\tau) \hat{\Lambda}(-\tau, -\tau-\tau') \tilde{V}(-\tau-\tau') \rangle \quad (82c)$$

$$\langle \tilde{K} \rangle = \int_0^\tau d\tau' \cdot \langle \tilde{V}(-\tau) \hat{U}(-\tau, -\tau-\tau') \tilde{V}(-\tau-\tau') \rangle . \quad (82d)$$

Now by neglecting the secondary memory, we can write the approximations:

$$\bar{H} \approx \bar{H} , \quad \bar{J} = \bar{J} \quad (83a)$$

$$\langle \tilde{K}(\tau) \rangle \{ \} = \langle \tilde{K}(\tau) \rangle . \quad (83b)$$

In the following, the determination of \hat{U} and \bar{U} will be performed through \hat{p} and \bar{p} , so that we will have no more opportunity of dealing with the kinetic equation (57) of the propagator.

VI. DETERMINATION OF THE COLLISIONS BY MEANS OF THE TRANSITION PROBABILITY

The result (79c) has transformed the convolution $\hat{U} * G$ of the Lagrangian form into its Fourier Eulerian correspondent. We shall apply this result to transform the convolution $\bar{H} * C_0$, rewritten as

$$\bar{H} * C_0 = \langle \Delta' \rangle \bar{U} * C_0 ,$$

by definition (53a). We obtain:

$$\begin{aligned}
\bar{H} * C_0 &\equiv \pi \langle \Delta' \rangle \int_0^\infty d\tau e^{i(\omega - \underline{k} \cdot \underline{v})\tau} \bar{P}(-\tau, \underline{k}) C_0(\omega, \underline{k}) \\
&= -\pi \langle \Delta' \rangle k^2 M(\omega, \underline{k}) C_0(\omega, \underline{k}) \\
&= -\alpha(\omega, \underline{k}) C_0(\omega, \underline{k}) \quad ,
\end{aligned} \tag{84}$$

where

$$\alpha(\omega, \underline{k}) \equiv \pi \langle \Delta' \rangle k^2 M(\omega, \underline{k}) \tag{85a}$$

is a dimensionless factor of memory, and

$$M(\omega, \underline{k}) = -k^{-2} \delta^2 \int_0^\infty d\tau e^{i(\omega - \underline{k} \cdot \underline{v})\tau} \bar{P}(-\tau, \underline{k}) \tag{85b}$$

is called the "memory function", defining a life-time $M^{1/3}$ for the memory.

Rigorously speaking, $\langle \Delta' \rangle \equiv \underline{\delta} \cdot \underline{(\Delta')} \cdot \underline{\delta}$ operates on the ballistic orbit function (15), as well as on all the diffusivities which are embedded in \bar{U} and C_0 . With the approximation that $\langle \underline{\Delta'} \rangle$ varies slowly with \underline{v} , as is true in strong turbulence, we can evaluate (85b) and get

$$M(\omega, \underline{k}) \cong \int_0^\infty d\tau \tau^2 e^{i(\omega - \underline{k} \cdot \underline{v})\tau} \bar{P}(-\tau, \underline{k}), \tag{86a}$$

where, by definition (76b), we have

$$\begin{aligned}
\bar{p}(-\tau, \underline{k}) &\equiv \langle \exp i \underline{k} \cdot \tilde{\underline{l}}(-\tau) \rangle \\
&= \int d\underline{l} \ e^{i \underline{k} \cdot \underline{l}} \bar{p}(-\tau, \underline{l}) \\
&= (2\pi)^d \bar{p}(-\tau, \underline{k}) .
\end{aligned} \tag{86b}$$

Now it leaves us the function $\bar{p}(-\tau, \underline{k})$, and we shall determine it by means of the Fourier transform of the kinetic equation of transition in the form

$$\frac{\partial}{\partial \tau} \bar{p}(-\tau, \underline{k}) = \bar{j}(-\tau, \underline{k}) \tag{87}$$

from (80a). The collision is given by the expression

$$\bar{j}(-\tau, \underline{k}) = -k^2 \langle \tilde{K}(\tau) \rangle \bar{p}(-\tau, \underline{k}) , \tag{88}$$

as governed by the diffusivity (82c). The integration with the condition (74b) gives the probability

$$\bar{p}(-\tau, \underline{k}) = (2\pi)^{-d} \exp \left[-k^2 \int_0^\tau d\tau' \langle \tilde{K}(\tau') \rangle \right] \tag{89a}$$

which we substitute into (86b) to obtain

$$\bar{p}(-\tau, \underline{k}) = \exp \left[-k^2 \int_0^\tau d\tau' \langle \tilde{K}(\tau') \rangle \right] . \tag{89b}$$

Since $\langle \tilde{K} \rangle$ is caused by the field fluctuations

$$\underline{\tilde{E}} = \underline{E}^0 + \underline{E}' \quad \text{and} \quad \underline{E}' = \underline{E}^{(1)} + \underline{E}'' ,$$

we can decompose the diffusivity into components, as follows:

$$\langle \tilde{K} \rangle = \langle K^0 \rangle + \langle K' \rangle, \quad \langle K' \rangle = \langle K^{(1)} \rangle + \langle K'' \rangle. \quad (90)$$

Within the available time span τ and under the quasi-stationarity condition (58), the diffusivity $\langle K'(\tau \rightarrow \infty) \rangle$ reaches a matured development, leaving the non-asymptotic diffusivities of the order of magnitude:

$$\langle K^{(1)}(\tau) \rangle \ll \langle K^0(\tau) \rangle.$$

The negligible $\langle K^{(1)}(\tau) \rangle$ may be included under $\langle K'(\tau) \rangle$ for brevity.

The procedures indicated are to calculate the two diffusivities from the path dynamics (66b), to substitute the results into (89b), and subsequently into (85a) and (85b), to obtain \bar{P} , α and M . The calculations are lengthy, but can be simplified if we choose the highest power in τ for retaining the most rapid memory-loss. The results are collected as follows:

(a) results relating to $\langle D' \rangle$

$$\bar{H} * C_0 \doteq -\alpha(\omega, k) C_0(\omega, k)$$

$$\bar{P}(\tau, k) \cong e^{-\omega_D'^3 \tau^3 - m^0 \tau^4}, \quad \tau = t - t'$$

$$\alpha(\omega, k) \cong c_\alpha \omega_D'^3 M(\omega, k), \quad c_\alpha = \pi/c' \quad (91)$$

$$M(\omega, k) \cong \int_0^\infty d\tau \tau^2 e^{i(\omega - k \cdot v)\tau} e^{-\omega_D'^3 \tau^3 - m^0 \tau^4}$$

with

$$\omega_D' = (c' \langle D' \rangle k^2)^{1/3}, \quad m^0 = (c' \langle (E^0)^2 \rangle k^2)^{1/4}, \quad c' = (1/3)^{1/3}, \quad c^0 = (1/8)^{1/4}$$

(b) results relating to $\langle D' \rangle$

$$\begin{aligned}\bar{H} * C_o &\equiv -\alpha(\omega, \underline{k}) C_o(\omega, \underline{k}) \\ \alpha(\omega, \underline{k}) &\cong c_s \omega_D'^3 M(\omega, \underline{k}) . \\ M(\omega, \underline{k}) &\cong \int_0^\infty d\tau \tau^2 e^{i(\omega - \underline{k} \cdot \underline{v})\tau} e^{-\omega_D'^3 \tau^3 - m^{*4} \tau^4}\end{aligned}\tag{92}$$

with

$$\omega_D' = (c_s \langle D' \rangle k^2)^{1/3}$$

By applying these results, we transform (53b) and (53c) into:

$$C_o(\omega, \underline{k}) = \underline{C}_o(\omega, \underline{k}) + \alpha(\omega, \underline{k}) \underline{C}_o(\omega, \underline{k})\tag{93}$$

$$\bar{H}(\omega, \underline{k}) = \bar{\underline{H}}(\omega, \underline{k}) + \alpha(\omega, \underline{k}) \bar{\underline{H}}(\omega, \underline{k}) .\tag{94a}$$

The last equation can be used to derive

$$\alpha(\omega, \underline{k}) = \underline{\alpha}(\omega, \underline{k}) + [\underline{\alpha}(\omega, \underline{k})]^2 .\tag{94b}$$

VII. SHIELDING AND ENHANCEMENT OF THE COLLISION

With the approximation of neglecting the secondary memory, as stated in (83), we can reduce (93) into the form

$$C_o(\omega, \underline{k}) \cong \underline{C}_o(\omega, \underline{k}) + \underline{\alpha}(\omega, \underline{k}) \underline{C}_o(\omega, \underline{k}) ,$$

rewritten as

$$C_o(\omega, \underline{k}) = \beta(\omega, \underline{k}) \underline{C}_o(\omega, \underline{k}) .\tag{95a}$$

by introducing the coefficient

$$\begin{aligned}\beta(\omega, \underline{k}) &= \left| (1 - \alpha(\omega, \underline{k}))^{-1} \right| \\ &= \left[(1 - \alpha_1)^2 + \alpha_2^2 \right]^{-1},\end{aligned}\tag{95b}$$

where α_1, α_2 are the real and imaginary parts of

$$\alpha = \alpha_1 + i \alpha_2.\tag{95c}$$

The result (95a) shows that the collective collision depends on the factor β , which in its turn depends on the memory function M . This function has a complicate integral (92). We shall calculate it approximately by means of an interpolation which correctly covers the three regions as dominated by ω'_D , m^0 , and $\omega - \underline{k} \cdot \underline{v}$, separately.

For the sake of simplification of writing, we introduce the following frequencies:

$$\gamma_{Dm} = \gamma_D + \gamma_m\tag{96a}$$

$$\gamma_m = \ell_m m^0, \quad \gamma_D = \ell_D \omega'_D, \quad \hat{\gamma}_D = \hat{\ell}_D \omega'_D, \quad \Omega = |\omega - \underline{k} \cdot \underline{v}|,$$

with the ratios

$$\xi = \Omega / \gamma_{Dm}, \quad \eta = \hat{\gamma}_D / \gamma_{Dm},\tag{96b}$$

and the numerical coefficients found to be:

$$\ell_m = \left[6 / \Gamma\left(\frac{7}{4}\right) \right]^{\frac{1}{3}}, \quad \ell_D = 6^{\frac{1}{3}}, \quad \hat{\ell}_D = (2c_s)^{\frac{1}{3}}.\tag{96c}$$

Finally we find the results, as follows:

$$\alpha_1 = \gamma^3 \frac{1-3\xi^2}{(1-3\xi^2)^2 + \xi^6(1+3\xi^{-2})^2} \cong \gamma^3(1+\xi^6)^{-1} \quad (97)$$

$$\alpha_2 = \xi^3 \alpha_1$$

We estimate that γ is of the order of unity, and ξ is an index of the strength of turbulence. The asymptotic cases reduce the general formula into the following:

(a) For weak turbulence, we have $\xi \gg 1$, entailing $\alpha_1 \ll \alpha_2$, we find

$$\alpha_2 \cong (\hat{\gamma}_D / \Omega)^3 \quad (98a)$$

so that

$$\beta = (1 + \alpha_2^2)^{-1} \quad (98b)$$

causing a shielded collision.

(b) For strong turbulence, we have $\xi \ll 1$, entailing $\alpha_2 \ll \alpha_1$, we find

$$\alpha_1 \cong (\hat{\gamma}_D / \gamma_{Dm})^3 = \left[\frac{\hat{\gamma}_D \omega'_D}{\gamma_D \omega'_D + \gamma_m m^0} \right]^3 < 1 \quad (99a)$$

so that

$$\beta \cong (1 - \alpha_1)^{-2} \quad (99b)$$

causing an enhanced collision.

The enhanced collision by the factor β can be evaluated for plasma turbulence and fluid turbulence. As an illustration, we consider the plasma turbulence in a strong magnetic field. Here the electric field fluctuation excites the velocity fluctuation. We found that both spectra possess the k^{-3} power law, that ω_D' is a constant quantity, and that the Larmor radius is a spectral cutoff. Since the E -fluctuation drops rapidly, the macro-electric energy $\langle (E^0)^2 \rangle$ will cover the main body of the energy-containing portion of the spectrum to become approximately independent of k . These parameters determine a turbulent Reynolds number

$$R_e = \langle (E^0)^2 \rangle^{1/2} k_c / \omega_D'^2 . \quad (100)$$

By definition (96a), we have

$$\gamma_m / \gamma_D = A R_e^{\frac{1}{2}} (k/k_c)^{\frac{1}{2}}, \quad A = \ell_m / \ell_D ,$$

obtaining

$$\alpha_1 = \left(\hat{\ell}_D / \ell_D \right)^3 \left[1 + A R_e^{\frac{1}{2}} (k/k_c)^{\frac{1}{2}} \right]^{-3}$$

and

$$\beta \approx 1 + 2 \left(\hat{\ell}_D / \ell_D \right)^3 \left[1 + A R_e^{\frac{1}{2}} (k/k_c)^{\frac{1}{2}} \right]^{-3} . \quad (101a)$$

For high intensity turbulence, R_e is large, and we reduce (101a) into:

$$\beta \approx 1 + c_\beta R_e^{-3/2} (k/k_c)^{-3/2} , \quad (101b)$$

with

$$c_{\beta} = 2 \left(\hat{\ell}_D / \ell_m \right)^3 .$$

The collective collision is increased above the pair collision by an amount proportional to $k^{-3/2}$. This means that the big eddies in a bath of smaller ones will be more apt to have an enhanced collective collision. An analogous increase of the effective diffusivity of a suspension of small particles in Brownian movements is well known. The application to fluid turbulence will be given at a later opportunity.

VIII. SUMMARY AND DISCUSSIONS

The point of departure of the statistical theory of turbulence is the description of the microdynamical state of turbulence, either by a system of hydrodynamical equations in the t, x_i space, e.g. the Navier-Stokes equations (16), or by a consistent microkinetic equation in the t, x_i, v_i space, e.g. the Liouville equation in the form (18). Here by enlarging the dimensionality from the passage of the t, x_i space to the t, x_i, v_i space, we have eliminated the nonlinearity as connected with the velocity field, but have kept the nonlinearity as connected with the E_i -field from (21).

If the Navier-Stokes equations are further decomposed into Fourier series, the multi-dimensional variable u_{α} , with $\alpha \equiv (i, k)$, has the three directions $i=1,2,3$ and all the Fourier modes k running from $-\infty$ to ∞ . The distribution

$$f(t, v_{\alpha}) = \delta \left[v_{\alpha} - u_{\alpha}(t) \right] \quad (102)$$

transforms the Navier-Stokes equation into a corresponding Liouville

equation of the form:

$$(\partial_t + L)f = 0, \quad (103)$$

where

$$L = - \sum_{\alpha} v_{\alpha} + \sum_{\alpha\beta\gamma} \frac{\partial}{\partial v_{\alpha}} A_{\alpha\beta\gamma} v_{\beta} v_{\gamma} \quad (104)$$

is the differential operator, and v_{α} , $A_{\alpha\beta\gamma}$ are the coefficients.

In this kinetic representation, the enlarged dimensionality by the Fourier transform renders L deterministic, so that the Liouville equation becomes linear. Since the linear equation does not distinguish between the average and the fluctuations, it does not lend to clearly express the coupling between the modes in a functional form. An external fluctuation response assumption becomes necessary for establishing the physical and functional structure of L . For example, when L and f are decomposed into two components as

$$L = (L)_0 + \delta L, \quad f = (f)_0 + \delta f, \quad (105)$$

(103) can be transformed into

$$\left[\partial_t + (L)_0 \right] (f)_0 = - \langle \delta L \delta f \rangle. \quad (106)$$

An iteration along the direct interaction approximation was to assume a dissipative $(L)_0$ of the Fokker-Planck form.

By contrast, the nonlinear Liouville equation (1a) retains the mode-couplings which are described by the equations of evolution (8a)-(8c). The form (8a) gives the collision in the most direct manner but requires the knowledge

of the operator $\hat{\Lambda}$ whose physical structure and relation to \hat{U} can only be obtained through other two equations, (8b) or (8c). In the form (8b), the memory disappears in the ballistic orbit and reappears as a high order correlation

$$\langle \tilde{L}(t) U_{\text{free}}(t, t-\tau) \tilde{L}(t-\tau) \tilde{f}(t-\tau) \rangle , \quad (107)$$

causing a hierarchy. Therefore we have chosen the form (8c) which does not present this difficulty. On the contrary it establishes the necessary relation between $\hat{\Lambda}$ and \hat{U} , thanks to the transition probability.

For the sake of simplification of discussion and abbreviation, we use the one-scale average by writing:

$$\bar{f}(t, \underline{x}, \underline{v}) = \bar{f}(1), \quad \tilde{f}(t, \underline{x}, \underline{v}) = \tilde{f}(1) , \quad (108a)$$

and

$$\tilde{L}(t, \underline{x}, \underline{v}) \tilde{f}(t, \underline{x}, \underline{v}) = \tilde{f}(1') \tilde{f}(1) , \quad (108b)$$

with

$$\tilde{f}(1') = \mathcal{E} \{ \tilde{f} \} , \quad (108c)$$

from (22b). The time derivatives along the trajectories will be written as:

$$\hat{d}_t = \partial_t + \hat{L} , \quad \bar{d}_t = \partial_t + \bar{L} . \quad (108d)$$

The average distribution and propagator, and their fluctuations, are governed by the following equations of evolution:

$$\bar{d}_t \bar{f}(1) = \bar{C}(1) \quad (109a)$$

$$\hat{d}_t \tilde{f}(1) = -\tilde{f}(1') \bar{f}(1) - \bar{c}(1) \quad (109b)$$

and

$$\bar{d}_t \bar{u}(1) = \bar{h}(1) \quad (110a)$$

$$\hat{d}_t \tilde{u}(1) = -\tilde{f}(1') \bar{u}(1) - \bar{h}(1), \quad (110b)$$

with the collisions:

$$\bar{c}(1) = -\langle \tilde{f}(1') \tilde{f}(1) \rangle \quad (111a)$$

$$\bar{h}(1) = -\langle \tilde{f}(1') \tilde{u}(1) \rangle. \quad (111b)$$

The evolution of $\tilde{f}(1)$ is governed by a variable source coupling \tilde{f} and \bar{f} and a deterministic source $-\bar{c}$ which represents the cluster of fluctuations in the diffusive medium. Upon integrating (109b), we get

$$\tilde{f}(1) = -\hat{\chi} \hat{U} * [\tilde{f}(1') \bar{f}(1) + \bar{c}(1)] , \quad (112)$$

and subsequently upon multiplying by $\tilde{f}(1')$ and averaging, we find the collision, as follows:

$$\bar{c}(1) = \langle \tilde{f}(1') \hat{\chi} \hat{U} * \tilde{f}(1') \rangle \bar{f}(1) + \langle \tilde{f}(1') \tilde{u} \rangle * \bar{c}(1) , \quad (113a)$$

or

$$\bar{c} = \bar{c} - \bar{h} * \bar{c} . \quad (113b)$$

Here

$$\bar{c} \equiv \langle \tilde{\Delta} \rangle \{ \bar{f} \} \quad (114)$$

is the pair collision, having an operator

$$\langle \tilde{\Delta} \rangle \equiv \langle \tilde{f}(1') \tilde{A} \hat{U} * \tilde{f}(1') \rangle, \quad (115)$$

and $-\bar{H} * \bar{C}$ is the convolution of two collisions:

$$\bar{C} = \langle \tilde{\Delta} \rangle \{ \bar{f} \} \quad \text{and} \quad \bar{H} = \langle \tilde{\Delta} \rangle \cdot \bar{U}. \quad (116)$$

In terms of the collision operator

$$\langle \tilde{\Delta} \rangle \equiv \langle \tilde{f}(1') \tilde{A} \hat{\Lambda} * f(1') \rangle, \quad (117)$$

we can rewrite (113b) as

$$\bar{C} = \langle \tilde{\Delta} \rangle \{ \bar{f} \} - \langle \tilde{\Delta} \rangle \{ \bar{U} \} * \langle \tilde{\Delta} \rangle \{ \bar{f} \}. \quad (118)$$

We conclude that the variable source and the deterministic sources in (109b) yield a collective collision with two components: one is proportional to $\langle \tilde{\Delta} \rangle$, and the other is proportional to $\langle \tilde{\Delta} \rangle^2$, as related to the pair collision and multiple collision, respectively. The latter contributes to a shielding of the collision in a weak turbulence as governed by the resonance of the wave $\exp [-i(\omega - \mathbf{k} \cdot \mathbf{v})\tau]$, or to an enhancement of the collision in a strong turbulence as governed by the diffusion parameters of relaxation frequencies ω'_D and m^0 in the memory function. Such opposite effects can find their analogies in the Balescu-Lenard equations in quiescent plasmas^{13,14} and in the effective diffusivity of a suspension of particles undergoing Brownian movements,¹⁵ respectively. More recently, a reversal of the effect of multiple collision in Brownian motion has also been found, dependent on the relative scales of the particles and the Brownian motion.¹⁶⁻¹⁸

We can give the following interpretation. A cluster of large size a may move with a high velocity and be scattered by the small scale turbulence, i.e. $ka > 1$. Since a part of the eddies will be displaced or shielded from participation in the collection interaction, the effective diffusivity is reduced. On the other hand, if $ka < 1$, the cluster will participate together with the background eddies to contribute an enhanced collision.

By the generalization to the two-scale averages, we obtain the kinetic equation of the distribution function f_0 with the collective collision C_0 , as found in (30) and (48). This can derive the transfer function for the energy-cascade and determine the spectrum of turbulence without the intermediary of the pair distribution function $\langle \tilde{f}(1') \tilde{f}(1) \rangle$.¹⁰

ACKNOWLEDGMENT. The author C. M. Tchen wishes to thank the Département de Physique du Plasma, Centre d'Etudes Nucléaires, for the hospitality which he enjoyed during his visit. His effort in this work forms part of the research sponsored by the NASA Grant No. NAS8-34622.

REFERENCES

- ¹J. Weinstock, Phys. Fluids 12, 1045 (1969).
- ²J. H. Misguich and C. M. Tchen, "Equivalent Methods for Describing Quasi-Linear Turbulent Trajectories". Report EUR-CEA-FC-1157 (1982).
- ³J. H. Misguich and C. M. Tchen, "A New Kinetic Description for Turbulent Collisions Including Mode-Coupling." Report EUR-CEA-FC-1158 (1982).
- ⁴T. Boutros-Ghali and T.H. Dupree, Phys. Fluids 24 (10), 1839 (1981).
- ⁵C. M. Tchen, Comptes Rendus Acad. Sci. Paris, 290B, 167 (1980).
- ⁶T. H. Dupree, Phys. Fluids 9, 1773 (1966).
- ⁷S. A. Orszag and R. H. Kraichnan, Phys. Fluids 10, 1720 (1967).
- ⁸C. M. Tchen, Phys. Fluids 16, 13 (1973).
- ⁹C. M. Tchen, Verslagen Ned. Akad. Wetensch. Afd. Natuurkunde 53, 400(1944).
- ¹⁰C. M. Tchen, "Kinetic Method of Turbulence". Preprint.
- ¹¹C. M. Tchen, Phys. Rev. A8, 500 (1973).
- ¹²C. M. Tchen, Plasma Phys. 18, 609 (1976).
- ¹³A. Lenard, Ann. Phys. (N.Y.) 3, 390 (1960).
- ¹⁴R. Balescu, Phys. Fluids 3, 52(1960).
- ¹⁵J. M. Burgers, Proc. Acad. Sci. Amsterdam 44, 1045, 1177 (1941).
- ¹⁶A. R. Altenberger, J. Chem. Phys. 70 (05), 1994(1979).

¹⁷W. B. Russel and A. B. Glendinning, J. Chem. Phys. 74(2), 928(1981).

¹⁸W. B. Russel, Ann. Rev. Fluid Mech. 13, 425(1981).

| | | | | | |
|--|--|--|---|---|--|
| 1. REPORT NO. NASA CR- 3787 | | 2. GOVERNMENT ACCESSION NO. | | 3. RECIPIENT'S CATALOG NO. | |
| 4. TITLE AND SUBTITLE Theory and Modeling of Atmospheric Turbulence Part I: September 1, 1981 - August 31, 1982 | | | | 5. REPORT DATE February 1984 | |
| | | | | 6. PERFORMING ORGANIZATION CODE | |
| 7. AUTHOR(S) C. M. Tchen | | | | 8. PERFORMING ORGANIZATION REPORT # | |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS The City College Research Foundation 138th Street and Convent Avenue New York, New York 10031 | | | | 10. WORK UNIT NO. M-440 | |
| | | | | 11. CONTRACT OR GRANT NO. NAS8-34622 | |
| | | | | 13. TYPE OF REPORT & PERIOD COVERED Contractor Report Sept 1, 1981 - Aug 31, 1982 | |
| 12. SPONSORING AGENCY NAME AND ADDRESS National Aeronautics and Space Administration Washington, D. C. 20546 | | | | 14. SPONSORING AGENCY CODE | |
| | | | | | |
| 15. SUPPLEMENTARY NOTES Technical Monitors: George H. Fichtl and Margaret B. Alexander Prepared for the Atmospheric Sciences Division, Systems Dynamics Laboratory Marshall Space Flight Center, Alabama 35812 Part I of Final Report | | | | | |
| 16. ABSTRACT Among the transport functions which characterize the evolution of a turbulent spectrum, the cascade transfer is the only function which describes the mode-coupling as the result of the nonlinear hydrodynamic state of turbulence. A kinetic theory combined with a scaling procedure is developed. The transfer function governs the nonlinear mode-coupling in strong turbulence. It is investigated by a kinetic theory of turbulence. The master equation is consistent with the hydrodynamical system that describes the micro-dynamical state of turbulence and possesses the advantages of being homogeneous and having fewer nonlinear terms. Since the detailed interactions between the individual modes contain too many minute details for a statistical theory, the modes are scaled into groups to decipher the governing transport processes and statistical characteristics. An equation of vorticity transport describes the micro-dynamical state of two-dimensional, isotropic and homogeneous, geostrophic turbulence. By group-scaling, the equation of evolution of the macro-vorticity is derived in the form of the Fokker-Planck equation with memory. The micro-dynamical state of turbulence is transformed into the Liouville equation to derive the kinetic equation of the singlet distribution in turbulence. The collision integral contains a memory, which is analyzed by considering the pair collision and the multiple collision. For the interactions among the groups, two other kinetic equations are developed in parallel for the propagator and the transition probability. | | | | | |
| 17. KEY WORDS Turbulence Atmospheric Boundary Layer Storm and Mesoscale Processes Global Scale Processes Atmospheric and Terrestrial Environ. Proc. | | | 18. DISTRIBUTION STATEMENT Unclassified-Unlimited STAR CATEGORY: 47 | | |
| 19. SECURITY CLASSIF. (of this report) Unclassified | | 20. SECURITY CLASSIF. (of this page) Unclassified | | 21. NO. OF PAGES 163 | |
| | | | | 22. PRICE A08 | |